SOCAR

SOCAR Proceedings

Fundamental scientific researches

journal home page: http://proceedings.socar.az

EMERGING SOURCES CITATION INDEX CLARIVATE ANALYTICS

Scopus

DEVELOPMENT OF DISCRETE ASYMPTOTIC ALGORITHM FOR THE OPTIMAL TRAJECTORY AND CONTROL IN OSCILLATORY SYSTEMS WITH LIQUID DAMPER

F. A. Aliev*[1,2], N. A. Aliev[1], A. F. Rasulzade[3], N. S. Hajiyeva[1], I. V. Alieva[4]

[1]*Institute of Applied Mathematics, Baku State University, Baku, Azerbaijan*

[2]*Institute of Information Technologies, Ministry of Science and Education of the Republic of Azerbaijan, Baku, Azerbaijan*

[3]*Azerbaijan Technical University, Baku, Azerbaijan*

[4]*Baku State University, Baku, Azerbaijan*

A B S T R A C T

In the current paper an asymptotic method to the problem of establishing the optimal program trajectory and optimal control in the movement of a sucker rod pumping unit of oscillatory systems with liquid damper is considered, where the plunger is inside the Newtonian fluid. In this case the mass of the head is large enough. From the properties of the plunger motion, the boundary conditions are taken to be periodic and the transition of motion from one mode to the second is described by impulse systems. By means of expedient transformations, the given equation of motion with fractional derivatives is reduced to the equation of fractional order containing a small parameter (the inverse of the mass of the head). Using the method of discretization of oscillatory systems with liquid dampers, a system of the first-order difference equations is reduced to the two-dimensional system. Using the given static data, the definition of fractional derivatives in subordinate terms is considered, a quadratic functional is constructed and this problem is investigated by the least squares method. Constructing the extended functional the discrete Euler-Lagrange equations are obtained. The control actions and the corresponding optimal program trajectory is found from the obtained system of discrete Euler-Lagrange equations using the Matlab software package and the algorithm of the calculation process is proposed. The results are illustrated with a specific, simple numerical example from practice and the graphs of optimal control and optimal program trajectory are given.

Keywords: asymptotic method; Newtonian fluid; Euler-Lagrange equations; algorithm of the calculation.

Date submitted: 01.04.2023 **Date accepted:** 29.05.2024

1. Introduction

In oil production one of the main methods is the procedure with a sucker rod pumping unit [1-6] of oscillatory systems with liquid damper, where the plunger is inside the Newtonian fluid. This allows describing linear ordinary differential equations [7-10] of the second order with fractional derivatives in subordinate numbers [11-13].

In the current paper asymptotic method of the oscillatory system with liquid dampers [14] for a sufficiently large head mass is given. In this case the corresponding problem is rather difficult, it makes sense to introduce a small parameter (by means of a rather heavy mass head) and construct an asymptotic solution in the first approximation. Introducing the functional and constructing the extended functional, a system of equations consisting of Euler-Lagrange equations [15-19] is established and the solution of the system is found.

2. Problem Statement of continuous case

We know that the motion equation of an oscillatory system in a liquid damper is determined by a fractional linear differential equation as follows [20-24]:

$$m_1\ddot{y}(x)+aD^{\alpha}y(x)+by(x)=f(x),\quad 0\le x<l \tag{1}$$

$$\begin{aligned} &y(l+0)=y(l-0),\\ &\dot{y}(l+0)=-\dot{y}(l-0)+V_1, \end{aligned} \tag{2}$$

$$m_2\ddot{y}(x)+aD^{\alpha}y(x)+by(x)=f(x),\quad l<x\le 2l, \tag{3}$$

$$\begin{aligned} &y(2l)=y(2l-0),\\ &\dot{y}(2l)=-\dot{y}(2l-0)+V_2, \end{aligned} \tag{4}$$

where $y(x)$ is a continuous function, a, b, V_1, V_2, m_1, m_2 are given $f(x)$ fixed numbers, and is a continuous scalar function, begin the force correspondig to the oscillatro system, a$\in(1,2)$. Let us assume a periodic boundary condition [1, 3, 4, 25] as follows:

$$\begin{cases} y(2l)=y(0),\\ \dot{y}(2l)=\dot{y}(0) \end{cases} \tag{6}$$

Note that we use the definition of $D^{\alpha}y(x)$ in Riemann-Liouville sense as

$$D^{\alpha}y(x)=\frac{d}{dx}\int_{x_0}^{x}\frac{(x-\tau)^{-\alpha}}{(-\alpha)!}y(\tau)d\tau.$$

**E-mail:* f_aliev@yahoo.com

http://dx.doi.org/10.5510/OGP20240200977

Dividing both sides of equation (3) by m_2 and denoting $\varepsilon=1/m_2$, we get

$$\ddot{y}(x)+\varepsilon a D^{\alpha} y(x)+\varepsilon b y(x)=\varepsilon f(x),\ \ l<x\le 2l, \tag{6}$$

where ε-is a small parameter.

It is required to find the minimum of the following quadratic functional

$$J=\frac{1}{2}\int_{x_0}^{l+x_0}\left(Y'(x)QY(x)+f^2(x)R\right)dx+ +\frac{1}{2}\int_{l+x_0}^{2l+x_0}\left(Y'(x,\varepsilon)QY(x,\varepsilon)+f^2(x)R\right)dx \tag{7}$$

where

$$Y(x)=\left[y(x)\ \ D^{\frac{1}{q}}y(x)\ \ D^{\frac{2}{q}}y(x)\ \dots\ \dot{y}(x)\ \dots\ D^{\frac{2q-1}{q}}y(x)\right]',$$

$$Y(x,\varepsilon)=\left[y(x,\varepsilon)\ \ D^{\frac{1}{q}}y(x,\varepsilon)\ \ D^{\frac{2}{q}}y(x,\varepsilon)\ \dots\ \dot{y}(x,\varepsilon)\ \dots\ D^{\frac{2q-1}{q}}y(x,\varepsilon)\right]'$$

$Q=Q'\geq 0$ is a symmetric matrix, $R>0$ is a scalar, the prime means the trnsposition operation

3. Discrete case

When discretizing equations (1), (6), we will get the following expressions, respectively [26, 27]:

$$W_{j+1}=A_jW_j+\psi_jW_0+F_j,\ \ j=\overline{0,m-1},\ \ i=\overline{m,2m-1}$$

$$W_j=\begin{pmatrix} y_{2j}\\ y_{2j+1}\end{pmatrix},$$

$$\psi_j=\left(\left(\sum_{k=1}^{m-2}\sum_{1<j_1<j_2<\dots<j_k\le m-2}\prod_{g=1}^{k}\psi^{j_{k+1-g}}\right)\psi^0+\psi^0\right),$$

$$\psi^{(m-1)}=\begin{pmatrix} A_{11}^{(m-1)} & A_{12}^{(m-1)}\\ A_{21}^{(m-1)} & A_{22}^{(m-1)}\end{pmatrix},$$

$$A_{11}^{(m-1)}=1-h\left[\frac{a}{m_1}\frac{(x_{2m-2}-x_{2m-4})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-2}-x_{2m-4})\right]+ +2h\left[\frac{a}{m_1}\frac{(x_{2m-2}-x_{2m-3})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-2}-x_{2m-3})\right],$$

$$A_{12}^{(m-1)}=2-h\left[\frac{a}{m_1}\frac{(x_{2m-2}-x_{2m-3})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-2}-x_{2m-3})\right],$$

$$\begin{aligned}A_{21}^{(m-1)}=&-2-2h\left[\frac{a}{m_1}\frac{(x_{2m-2}-x_{2m-4})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-2}-x_{2m-4})\right]+\\&+4h\left[\frac{a}{m_1}\frac{(x_{2m-2}-x_{2m-3})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-2}-x_{2m-3})\right]-\\&-h\left[\frac{a}{m_1}\frac{(x_{2m-1}-x_{2m-4})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-1}-x_{2m-4})\right]+\\&+2h\left[\frac{a}{m_1}\frac{(x_{2m-1}-x_{2m-3})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-1}-x_{2m-3})\right]+\\&h^2\left[\frac{a}{m_1}\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-1}-x_{2m-2})\right]\times\\&\times\left\{\begin{matrix}\left[\frac{a}{m_1}\frac{(x_{2m-2}-x_{2m-4})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-2}-x_{2m-4})\right]-\\ 2\left[\frac{a}{m_1}\frac{(x_{2m-2}-x_{2m-3})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-2}-x_{2m-3})\right]\end{matrix}\right\},\end{aligned}$$

$$\begin{aligned}A_{22}^{(m-1)}=&3-2h\left[\frac{a}{m_1}\frac{(x_{2m-2}-x_{2n-3})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-2}-x_{2m-3})\right]-\\&-h\left[\frac{a}{m_1}\frac{(x_{2m-1}-x_{2m-3})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-1}-x_{2m-3})\right]+\\&+h^2\left[\frac{a}{m_1}\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-1}-x_{2m-2})\right]\times\\&\times\left[\frac{a}{m_1}\frac{(x_{2m-2}-x_{2m-3})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-2}-x_{2m-3})\right],\end{aligned}$$

$$\psi^{(k)}=\begin{pmatrix} A_{11}^{(k)} & A_{12}^{(k)}\\ A_{21}^{(k)} & A_{22}^{(k)}\end{pmatrix}\qquad k=\overline{0,m-2},$$

$$\begin{aligned}A_{11}^{(k)}=&-2h\left\{\left[\frac{a}{m_1}\frac{(x_{2m-2}-x_{2k-2})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-2}-x_{2k-2})\right]-\right.\\&-2\left[\frac{a}{m_1}\frac{(x_{2m-2}-x_{2k-1})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-2}-x_{2k-1})\right]+\\&\left.+\left[\frac{a}{m_1}\frac{(x_{2m-2}-x_{2k})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-2}-x_{2k})\right]\right\},\\&k=\overline{1,m-2},\end{aligned}$$

$$A_{11}^{(0)}=-h\left[\frac{a}{m_1}\frac{(x_{2m-2}-x_0)^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-2}-x_0)\right],$$

$$\begin{aligned}A_{12}^{(k)}=&-h\left\{\left[\frac{a}{m_1}\frac{(x_{2m-2}-x_{2k-1})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-2}-x_{2k-1})\right]-\right.\\&-2\left[\frac{a}{m_1}\frac{(x_{2m-2}-x_{2k})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-2}-x_{2k})\right]+\\&\left.+\left[\frac{a}{m_1}\frac{(x_{2m-2}-x_{2k+1})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-2}-x_{2k+1})\right]\right\},\\&k=\overline{1,m-2},\end{aligned}$$

$$\begin{aligned}A_{12}^{(0)}=&2h\left[\frac{a}{m_1}\frac{(x_{2m-2}-x_0)^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-2}-x_0)\right]-\\&-h\left[\frac{a}{m_1}\frac{(x_{2m-2}-x_1)^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-2}-x_1)\right],\end{aligned}$$

$$\begin{aligned}A_{21}^{(k)}=&-h\left\{2-h\left[\frac{a}{m_1}\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-2}-x_{2m-2})\right]\right\}-\\&\left\{\left[\frac{a}{m_1}\frac{(x_{2m-2}-x_{2k-2})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-2}-x_{2k-2})\right]-\right.\\&-2\left[\frac{a}{m_1}\frac{(x_{2m-2}-x_{2k-1})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-2}-x_{2k-1})\right]+\\&\left.+\left[\frac{a}{m_1}\frac{(x_{2m-2}-x_{2k})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-1}-x_{2k})\right]\right\}-\\&-h\left[\frac{a}{m_1}\frac{(x_{2m-1}-x_{2k-2})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-1}-x_{2k-2})\right]-\\&-2\left[\frac{a}{m_1}\frac{(x_{2m-1}-x_{2k-1})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-1}-x_{2k-1})\right]+\\&\left.+\left[\frac{a}{m_1}\frac{(x_{2m-1}-x_{2k})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-1}-x_{2k})\right]\right\},\\&k=\overline{1,m-2},\end{aligned}$$

$$A_{21}^{(0)}=-h\left\{2-h\left[\frac{a}{m_1}\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-1}-x_{2m-2})\right]\right\}\times$$
$$\times\left[\frac{a}{m_1}\frac{(x_{2m-2}-x_0)^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-2}-x_0)\right]-$$
$$-h\left[\frac{a}{m_1}\frac{(x_{2m-1}-x_0)^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-1}-x_0)\right],$$

$$A_{22}^{(k)}(a,n,h)=$$
$$=-h\left\{2-h\left[\frac{a}{m_1}\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-1}-x_{2m-2})\right]\right\}\times$$
$$\times\left\{\left[\frac{a}{m_1}\frac{(x_{2m-2}-x_{2k-1})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-2}-x_{2k-1})\right]-\right.$$
$$-2\left[\frac{a}{m_1}\frac{(x_{2m-2}-x_{2k})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-2}-x_{2k})\right]+$$
$$+\left[\frac{a}{m_1}\frac{(x_{2m-2}-x_{2k+1})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-2}-x_{2k+1})\right]-$$
$$-h\left\{\left[\frac{a}{m_1}\frac{(x_{2m-1}-x_{2k-1})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-1}-x_{2k-1})\right]-\right.$$
$$-2\left[\frac{a}{m_1}\frac{(x_{2m-1}-x_{2k})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-1}-x_{2k})\right]+$$
$$+\left[\frac{a}{m_1}\frac{(x_{2m-1}-x_{2k-1})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-1}-x_{2k-1})\right]-$$
$$-2\left[\frac{a}{m_1}\frac{(x_{2m-1}-x_{2k})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-1}-x_{2k})\right]+$$
$$\left.\left.+\left[\frac{a}{m_1}\frac{(x_{2m-1}-x_{2k+1})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-1}-x_{2k+1})\right]\right\}\right\},$$
$$k=\overline{1,m-2},$$

$$A_{22}^{(0)}=h\left\{2-h\left[\frac{a}{m_1}\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-1}-x_{2m-2})\right]\right\}\times$$
$$\times\left\{2\left[\frac{a}{m_1}\frac{(x_{2m-2}-x_0)^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-2}-x_0)\right]\right\}-$$
$$-\left[\frac{a}{m_1}\frac{(x_{2m-2}-x_1)^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-2}-x_1)\right]+$$
$$+2h\left[\frac{a}{m_1}\frac{(x_{2m-1}-x_0)^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-1}-x_0)\right]-$$
$$-h\left[\frac{a}{m_1}\frac{(x_{2m-1}-x_1)^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m_1}(x_{2m-1}-x_1)\right],$$

$$F_j=\begin{pmatrix}f_{1j}\\ f_{2j}\end{pmatrix}\quad j=\overline{0,m-1}$$

$$f_{1j}=h^2F_{2m-2}$$
$$f_{2j}=(2h^2-h^3K_\alpha(x_{2m-1}-x_{2m-2}))F_{2m-2}+h^2F_{2m-1}$$

$$K_\alpha(x-t)=a\frac{(x-t)^{1-\alpha}}{(1-\alpha)!}+b(x-t),$$

$$W_{i+1}(\varepsilon)=\left(\Gamma+\varepsilon\tilde{A}^{i-1}\right)W_i+\Psi_i(\varepsilon)W_0+F_i(\varepsilon),$$
$$i=\overline{m,2m-1}$$

$$\Psi_m(\varepsilon)=\left(\left(\sum_{k=1}^{m-2}\sum_{1<i_1<i_2<\ldots<i_k\le m-2}\prod_{j=1}^{k}A^{i_{k+1-j}}(\varepsilon)\right)A^0(\varepsilon)+A^0(\varepsilon)\right),$$

where

$$A^{m-1}(\varepsilon)=\begin{pmatrix}-1-\varepsilon\tilde{A}_{11}^{m-1} & 2+\varepsilon\tilde{A}_{12}^{m-1}\\ -2+\varepsilon\tilde{A}_{21}^{m-1}+\varepsilon^2\tilde{\tilde{A}}_{21}^{m-1} & 3+\varepsilon\tilde{A}_{22}^{m-1}+\varepsilon^2\tilde{\tilde{A}}_{22}^{m-1}\end{pmatrix}=$$
$$=\begin{pmatrix}-1 & 2\\ -2 & 3\end{pmatrix}+\varepsilon\begin{pmatrix}-\tilde{A}_{11}^{m-1} & \tilde{A}_{12}^{m-1}\\ \tilde{A}_{21}^{m-1} & \tilde{A}_{22}^{m-1}\end{pmatrix}=\begin{pmatrix}-1 & 2\\ -2 & 3\end{pmatrix}+\varepsilon\tilde{A}^{m-1}$$
$$\Gamma=\begin{pmatrix}-1 & 2\\ -2 & 3\end{pmatrix}$$
$$\tilde{A}^{m-1}=\begin{pmatrix}-\tilde{A}_{11}^{m-1} & \tilde{A}_{12}^{m-1}\\ \tilde{A}_{21}^{m-1} & \tilde{A}_{22}^{m-1}\end{pmatrix},$$

$$\tilde{A}_{11}^{m-1}(\varepsilon)=hK_\alpha\left(x_{2m-2}-x_{2m-4}\right)+2hK_\alpha\left(x_{2m-2}-x_{2m-3}\right),$$
$$\tilde{A}_{12}^{m-1}(\varepsilon)=-hK_\alpha\left(x_{2m-2}-x_{2m-3}\right),$$
$$\tilde{A}_{21}^{m-1}=-2hK_\alpha\left(x_{2m-2}-x_{2m-4}\right)+4hK_\alpha\left(x_{2m-2}-x_{2m-3}\right)-$$
$$-hK_\alpha\left(x_{2m-1}-x_{2m-4}\right)+2hK_\alpha\left(x_{2m-1}-x_{2m-3}\right),$$
$$\tilde{A}_{22}^{m-1}=-2hK_\alpha\left(x_{2m-2}-x_{2m-3}\right)-hK_\alpha\left(x_{2m-1}-x_{2m-3}\right),$$
$$A^k(\varepsilon)=\begin{pmatrix}\varepsilon\tilde{A}_{11}^k & \varepsilon\tilde{A}_{12}^k\\ \varepsilon\tilde{A}_{21}^k+\varepsilon^2\tilde{\tilde{A}}_{21}^k & \varepsilon\tilde{A}_{22}^k+\varepsilon^2\tilde{\tilde{A}}_{22}^k\end{pmatrix}=$$
$$=\varepsilon\begin{pmatrix}\tilde{A}_{11}^k & \tilde{A}_{12}^k\\ \tilde{A}_{21}^k & \tilde{A}_{22}^k\end{pmatrix}=\varepsilon\tilde{A}^k(\varepsilon),$$
$$k=\overline{1,m-2}$$

$$\tilde{A}_{11}^k=-hK_\alpha\left(x_{2m-2}-x_{2k-2}\right)+2hK_\alpha\left(x_{2m-2}-x_{2k-1}\right)-$$
$$-hK_\alpha\left(x_{2m-2}-x_{2k}\right)$$
$$\tilde{A}_{12}^k=-hK_\alpha\left(x_{2m-2}-x_{2k-1}\right)+2hK_\alpha\left(x_{2m-2}-x_{2k}\right)-$$
$$-hK_\alpha\left(x_{2m-2}-x_{2k+1}\right)$$

$$\tilde{A}_{21}^k=-2hK_\alpha\left(x_{2m-2}-x_{2k-2}\right)+4hK_\alpha\left(x_{2m-2}-x_{2k-1}\right)-$$
$$-2hK_\alpha\left(x_{2m-2}-x_{2k}\right)-$$
$$-hK_\alpha\left(x_{2m-1}-x_{2k-2}\right)+2hK_\alpha\left(x_{2m-2}-x_{2k-1}\right)-$$
$$-hK_\alpha\left(x_{2m-1}-x_{2k}\right),$$

$$\tilde{A}_{22}^k=-2hK_\alpha\left(x_{2m-2}-x_{2k-1}\right)+4hK_\alpha\left(x_{2m-2}-x_{2k}\right)-$$
$$-2hK_\alpha\left(x_{2m-2}-x_{2k+1}\right)-hK_\alpha\left(x_{2m-1}-x_{2k-1}\right)+$$
$$+2hK_\alpha\left(x_{2m-2}-x_{2k}\right)-hK_\alpha\left(x_{2m-1}-x_{2k+1}\right),$$

$$A^0(\varepsilon)=\begin{pmatrix}\varepsilon\tilde{A}_{11}^0 & \varepsilon\tilde{A}_{12}^0\\ \varepsilon\tilde{A}_{21}^0+\varepsilon^2\tilde{\tilde{A}}_{21}^0 & \varepsilon\tilde{A}_{22}^0+\varepsilon^2\tilde{\tilde{A}}_{22}^0\end{pmatrix}=$$
$$=\varepsilon\begin{pmatrix}\tilde{A}_{11}^0 & \tilde{A}_{12}^0\\ \tilde{A}_{21}^0 & \tilde{A}_{22}^0\end{pmatrix}=\varepsilon\tilde{A}^0,$$
$$\tilde{A}^0=\begin{pmatrix}\tilde{A}_{11}^0 & \tilde{A}_{12}^0\\ \tilde{A}_{21}^0 & \tilde{A}_{22}^0\end{pmatrix},$$

$$\tilde{A}_{11}^0=-hK_\alpha\left(x_{2m-2}-x_0\right),$$
$$\tilde{A}_{12}^0=2hK_\alpha\left(x_{2m-2}-x_0\right)-hK_\alpha\left(x_{2m-2}-x_1\right),$$
$$\tilde{A}_{21}^0=-2hK_\alpha\left(x_{2m-2}-x_0\right)-hK_\alpha\left(x_{2m-1}-x_0\right),$$
$$\tilde{A}_{22}^0=4hK_\alpha\left(x_{2m-2}-x_0\right)+2hK_\alpha\left(x_{2m-2}-x_0\right)-$$
$$-hK_\alpha\left(x_{2m-1}-x_1\right)-2hK_\alpha\left(x_{2m-2}-x_1\right),$$

$$F_i=\begin{pmatrix}f_{1i}\\ f_{2i}\end{pmatrix}\quad i=\overline{m,2m-1}$$
$$f_{1i}=h^2F_{2m-2}$$
$$f_{2i}=(2h^2-h^3K_\alpha(x_{2m-1}-x_{2m-2}))F_{2m-2}+h^2F_{2m-1}$$

If we discretize conditions (2), (4), (5) and the functional (7) we get the following [25, 28, 29]:

$$(-1 \quad 1)W_m = 0$$
$$(1 \quad 0)\left(W_{m+1} - W_m\right) = V_1$$
$$(-1 \quad 1)W_{2m} = 0$$
$$(-1 \quad 0)W_0 + (0 \quad 1)W_{2m} = 0$$
$$(-1 \quad 1)(W_0 + W_{2m}) = V_2$$

$$J = \sum_{i=0}^{m-1} W_j' Q W_j + \sum_{j=m}^{2m-1} W_i' Q W_i + \sum_{i=0}^{2m-1} F_i' C F_i \to \min$$

where $Q=Q'\geq 0 \;\; 2m\times 2m$ is an arbitrary symmetric matrix of dimension, $C>0 \;\; 2m\times 2m$ i is an arbitrary matrix.

Thus, we get the following discrete problem similarly the problem (1)-(7)

$$W_{j+1} = A_j W_j + \psi_j W_0 + F_j, \quad j = \overline{0, m-1} \tag{8}$$

$$W_{i+1}(\varepsilon) = \left(\Gamma + \varepsilon \tilde{A}^{i-1}\right) W_i + \Psi_i(\varepsilon) W_0 + F_i(\varepsilon), \quad i = \overline{m, 2m-1} \tag{9}$$

$$\begin{gathered}(-1 \quad 1)W_m = 0\\ (1 \quad 0)\left(W_{m+1} - W_m\right) = V_1\\ (-1 \quad 1)W_{2m} = 0\\ (-1 \quad 0)W_0 + (0 \quad 1)W_{2m} = 0\\ (-1 \quad 1)(W_0 + W_{2m}) = V_2\end{gathered} \tag{10}$$

$$J = \sum_{i=0}^{m-1} W_j' Q W_j + \sum_{j=m}^{2m-1} W_i' Q W_i + \sum_{i=0}^{2m-1} F_i' C F_i \to \min \tag{11}$$

4. Finding a solution to the problem (8)-(11)

For this, let's construct an extended functional as follows [19]

$$\begin{aligned}\bar{J}(\varepsilon) &= \alpha\left((-1 \quad 1)W_m\right) + \beta\left((1 \quad 0)\left(W_{m+1} - W_m\right) - h \cdot V_1\right) + \\ &+ \gamma\left((-1 \quad 1)W_{2m}\right) + \xi\left((-1 \quad 0)W_0 + (0 \quad 1)W_{2m}\right) + \\ &+ \eta\left((-1 \quad 1)(W_0 + W_{2m}) - h \cdot V_2\right) + \\ &+ \frac{1}{2}\left(\sum_{j=0}^{m-1} W_j' Q W_j + F_i' C F + \lambda_{j+1}^T \left(A_j W_j + \psi_j W_0 + F_j - W_{j+1}\right) + \right. \\ &+ \sum_{i=m}^{2m-1}\left(W_i' Q W_i + F_i'(\varepsilon) C F_i(\varepsilon) \lambda_{i+1}^T \times \right. \\ &\times \left(\left(\Gamma + \varepsilon \tilde{A}^{i-1}\right) W_i + \Psi_i(\varepsilon) W_0 + F_i(\varepsilon) - W_{i+1}(\varepsilon)\right)\Big)\end{aligned}$$

where are α, β, γ, η, ξ scalar, λ_i-1×2 is a-dimensional olumn vector.

Then let's write the Euler Lagrange equations [19, 25, 30]:

$$\begin{cases} W_{j+1} = A_j W_j + \psi_j W_0 - C^{-1}\lambda_{j+1}, \quad j = \overline{0, m-1} \\ W_{i+1}(\varepsilon) = \left(\Gamma + \varepsilon \tilde{A}^{i-1}\right) W_i + \Psi_i(\varepsilon) W_0 - C^{-1}\lambda_{i+1}, \; i = \overline{m, 2m-1} \\ \lambda_1 = Q W_1 + \psi_j^T \lambda_2, \\ \lambda_2 = Q W_2 + \psi_j^T \lambda_3, \\ \vdots \\ \lambda_{m-1} = Q W_{m-1} + \psi_j^T \lambda_m, \\ \lambda_m = \alpha(-1 \quad 1)^T - \beta(1 \quad 0)^T + Q W_m(\varepsilon) + \left(\Gamma + \varepsilon \tilde{A}^m\right)^T \lambda_{m+1} \\ \lambda_{m+1} = \beta(1 \quad 0)^T + Q W_{m+1}(\varepsilon) + \left(\Gamma + \varepsilon \tilde{A}^{m+1}\right)^T \lambda_{m+2} \\ \lambda_{m+2} = Q W_{m+2}(\varepsilon) + \left(\Gamma + \varepsilon \tilde{A}^{m+1}\right)^T \lambda_{m+3} \\ \vdots \\ \lambda_{2m-1} = Q W_{2m-1}(\varepsilon) + \left(\Gamma + \varepsilon \tilde{A}^{2m}\right)^T \lambda_{2m} \\ \lambda_{2m} = \gamma(-1 \quad 1)^T + \xi(0 \quad 1)^T + \eta(-1 \quad 1)^T \end{cases}$$

$$\begin{pmatrix} W_{j+1} \\ \lambda_j \end{pmatrix} = \begin{pmatrix} A_j & -C^{-1} \\ Q & A_j^T \end{pmatrix} \begin{pmatrix} W_j \\ \lambda_{j+1} \end{pmatrix} + \begin{pmatrix} \psi_j W_0 \\ 0 \end{pmatrix}, j = \overline{0, m-1},$$

$$\begin{pmatrix} W_{i+1}(\varepsilon) \\ \lambda_i \end{pmatrix} = \begin{pmatrix} \Gamma + \varepsilon \tilde{A}^{i-1} & -C^{-1} \\ Q & \left(\Gamma + \varepsilon \tilde{A}^{i-1}\right)^T \end{pmatrix} \begin{pmatrix} W_i \\ \lambda_{i+1} \end{pmatrix} + \begin{pmatrix} \Psi_i(\varepsilon) W_0 \\ \alpha(-1 \quad 1)^T + \gamma(-1 \quad 1)^T + \xi(0 \quad 1)^T + \eta(-1 \quad 1)^T \end{pmatrix},$$
$$i = \overline{m, 2m-1}$$

We can propose the following algorithm to find the solution of the problem (8)-(11).

Algorithm

STEP 1: m_1, ε, n, h, α, Q, C, V_1, V_2 initially given.

STEP 2: $\psi_i, A_i, \Psi_i(\varepsilon), A_i(\varepsilon)$ they are calculated.

STEP 3: Boundary and periodicity conditions are established depending on n.

STEP 4: The functional (11) and extended functional the corresponding (11) are constructed.

STEP 5: The Euler-Lagrange equations are set up and calculated.

STEP 6: The solution to the problem is found using the Matlab software package.

5. Example

Let we have the parameters including the problem (8)-(11) [31] $m_1 = 100$, $\varepsilon = 10^{-4}$, $a = 1$, $b = 0$, $\alpha = 1.6995$, $m = 4$,

$Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $V_1 = 1$, $V_2 = 2$, $\Gamma = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$, $x \in (0,1]$, $l = 50$, $h = (1 - 0.1)/l$

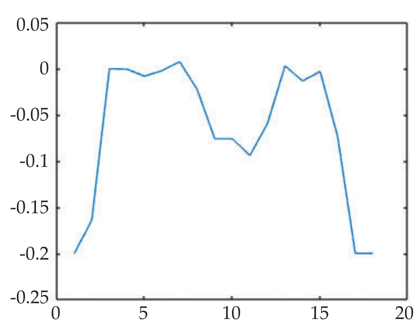

Fig.1 Depending y_m *on m.*

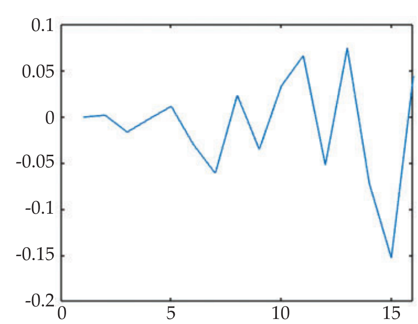

Fig.2. Depending F_m *on m.*

Conclusion

In the paper asymptotic method for defining the optimal program trajectory and control for discrete oscillatory systems with liquid dampers by periodic discrete boundary condition in sucker-rod pumping unit during oil production [32-34] using Euler-Lagrange method is constructed. At the end the numerical example is proposed, the graphs of optimal program trajectory and control have been shown.

References

1. Rasulzade, A. F., Hajiyeva, N. S., Ismailov, N. A. (2022). Determination of the optimal program trajectory and control for discretized movement of the plunger in rod pumping equipment. *Proceedings of IAM*, 11(2), 203-220.

2. Gurbanov, A. N., Sardarova, I. Z. (2022). Optimization problem of measurements in experimental research of gas-lift wells. *Applied and Computational Mathematics*, 21(2), 223-228.

3. Aliev, F. A., Aliev, N. A., Rasulzade, A. F., Hajiyeva, N. S. (2023). Solution of the optimal program trajectory and control of the discretized equation of motion of sucker-rod pumping unit in a Newtonian fluid. *TWMS Journal of Pure and Applied Mathematics*, 13(4), 1369-1382.

4. Hajiyeva, N. S. (2022). Determination of optimal program trajectory and control for the movement of the plunger sucker-rod pumping unit. *Proceedings of IAM*, 11(2), 137-158.

5. Jamalbayov, M. A., Valiyev, N. A. (2024). The discrete-imitational modeling concept of the «sucker-rod pump-well-reservoir» system and the optimization of the pumping process. *Petroleum Research*, In Press. https://doi.org/10.1016/j.ptlrs.2024.04.001

6. Jamalbayov, M. A., Veliyev, N. A. (2017). The technique of early determination of reservoir drive of gas condensate and velotail oil deposits on the basis of new diagnosis indicators. *TWMS Journal of Pure and Applied Mathematics*, 8(2), 236-250.

7. Andreev, Yu. I. (1976). Control of finite-dimensional linear objects. *Moscow: Nauka.*

8. Kalman, R., Falb, P., Arbib, M. (1972). Essays on the mathematical theory of systems. *Moscow: Mir.*

9. Musayeva, M. A., Hashimov, S. A. (2023). Identification problem for nonlinear schrodinger type equation. *TWMS Journal of Pure and Applied Mathematics*, 14(2), 162-177.

10. Suleimanov, B. A., Dyshin, O. A. (2013). Application of discrete wavelet transform to the solution of boundary value problems for quasi-linear parabolic equations. *Applied Mathematics and Computation*, 219, 7036-7047.

11. Aliev, F. A., Aliyev, N. A., Hajiyeva, N. S., Mahmudov, N. I. (2021). Some mathematical problems and their solutions for the oscillating systems with liquid dampers: review. *Applied and Computational Mathematics*, 20(3), 339-365.

12. Çelik, B., Set, E., Akdemir, A. O., Özdemir, M. E. (2023). Novel generalizations for Grüss type inequalities pertaining to the constant proportional fractional integrals. *Applied and Computational Mathematics*, 22(2), 275-291.

13. Aliev, F. A., Aliev, N. A., Hajiyeva, N. S., et al. (2021). Solution of an oscillatory system with fractional derivative including to equations of motion and to nonlocal boundary conditions. *SOCAR Proceedings*, 4, 115-121.

14. Aliev, F. A., Aliev, N. A., Hajiyeva, N. S., et al. (2022). Asymptotic method for solution of oscillatory fractional derivative. *Computational Methods for Differential Equations*, 10(4), 1123-1130.

15. Aliev, F. A., Hajiyeva, N. S., Ismailov, N. A., Mirsaabov, S. M. (2019). Calculation algorithm defining the coefficient of hydraulic resistance on different areas of pump-compressor pipes in gas lift process by lines method. *SOCAR Proceedings*, 4, 13-17.

16. Aliev, F. A., Larin, V. B. (1998). Optimization of linear control systems: analytical methods and computational algorithms. *Amsterdam: Gordon and Breach Sci.*

17. Aliev, F. A., Larin, V. B., Velieva, N. I. (2022). Algorithms of the synthesis of optimal regulators. *USA: Outskirts Press.*

18. Aliev, F. A. (1989). Methods for solving applied problems of optimization of dynamic systems. *Baku: Elm.*

19. Bryson, A., Ho, Yu. Sh. (1972). Applied theory of optimal control. *Moscow: Mir.*

20. Allaberen, A., Mesut, U. (2022). Stability of time-dependent source identification problem for Schrodinger differential equations. *TWMS Journal of Pure and Applied Mathematics*, 13(2), 245-255.

21. Dolicanin, C., Kacapor, E., Atanackovic, T. M. (2023). A note on the heavy longest reach cantilever: An unconventional optimization problem. *Applied and Computational Mathematics,* 22(4), 466-479.

22. Monje, C. A., Chen, Y. Q, Vinagre, B. M, et al. (2010). Fractional – order systems and controls fundamentals and applications. *London: Springer.*

23. Odibat, Z. M. (2010). Analytic study on linear systems of fractional differential equations. *Computers & Mathematics with Applications,* 59(3), 1171-1183.

24. Samko, S. G., Kilbas, A. A., Marichev, O. I. (1993). Fractional integrals and derivatives: Theory and applications. *Yverdon, Switzerland: Gordon and Breach Science Publishers.*

25. Mukhtarova, N. S., Ismailov, N. A. (2014). Algorithm to solution of the optimization problem with periodic condition and boundary control. *TWMS Journal of Pure and Applied Mathematics*, 5(1), 130-137.

26. Aliev, N. A., Velieva, N. I., Gasımova, K. G., Resulzade, A. F. (2019). Discretization method on movement equation of the oscillating system with liquid dumpers. *Proceedings of IAM*, 8(2), 211-228.

27. Rasulzada, A. F., Aliev, N. A., Velieva, N. I. (2021). Algorithm for determining fractional derivatives for discrete vibration systems with a liquid damper. *Proceedings of IAM*, 10(2), 181-191.

28. Xu, C. J., Lin, J., Zhao, Y., et al. (2023). Hopf bifurcation control of a fractional-order delayed turbidostat model via a novel extended hybrid controller. *Applied and Computational Mathematics*, 22(4), 495-519.

29. Hamidov, S. I. (2022). Optimal trajectories in reproduction models of economic dynamics. *TWMS Journal of Pure and Applied Mathematics*, 13(1), 16-24.

30. Shakhmurov, V., Maharramov, A., Sahmurova, A. (2023). Stability features of the dynamical system emerging in the model of the cancer growth. *TWMS Journal of Pure and Applied Mathematics*, 14(1), 23-40.

31. Bonilla, B., Rivero, M., Trujillo, J. J. (2007). On systems of linear fractional equations with constant coefficients. *Applied Mathematics and Computation*, 187(1), 68-76.

32. Suleimanov, B. A., Feyzullayev, Kh. A., Abbasov, E. M. (2019). Numerical simulation of water shut-off performance for heterogeneous composite oil reservoirs. *Applied and Computational Mathematics*, 18(3), 261-271.

33. Suleimanov, B. A., Ismailov, F. S., Dyshin, O. A. (2014). Statistical modeling of life cycle of oil reservoir development. *Journal of the Japan Petroleum Institute*, 57(1), 47-57.

34. Suleimanov, B. A., Feyzullayev, Kh. A. (2024). Simulation study of water shut-off treatment for heterogeneous layered oil reservoirs. *Journal of Dispersion Science and Technology*, Published online 6 Apr 2024.