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GENERALIZED LOCAL MORREY REGULARITY OF ELLIPTIC SYSTEMS

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A B S T R A C T

We study the regularity theory of linear elliptic systems with dis- continuous coefficients in generalized local Morrey spaces. Precisely, we obtain local regularity results for the strong solutions to 2*b*-order linear elliptic systems

$$L(x,D)\mathbf{u} := \sum_{|\alpha|=2b} \mathbf{A}_\alpha(x) D^\alpha \mathbf{u}(x) = \mathbf{f}(x)$$

where the principal coefficients $\mathbf{A}_\alpha$ are assumed to be functions with vanishing mean oscillation (*VMO*).

Keywords: elliptic systems; generalized local morrey space; vanishing mean oscillation.

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1. Introduction

The classical Morrey spaces $L_{p\lambda}$ are originally introduced in [1] in order to study the local behavior of solutions to elliptic partial differential equations. In fact, the better inclusion between the Morrey and the Holder spaces permits to obtain higher regularity of the solutions to different elliptic and parabolic boundary problems. The first author, Mizuhara and Nakai [2 -4] introduced generalized Morrey spaces $M_{p\varphi}$ independently (see, also [5, 6]). In general, Garcia-Cuerva and Herrero [7] and the first author [2] local Morrey spaces also introduced independently (see also [8). Note that, in [2] the first author introduced the local Morrey-type spaces including the generalized local Morrey spaces and studied integral operators in these spaces.

We deal with local regularity results and derive interior a priori estimates in generalized local Morrey spaces for the strong solutions of the uniformly elliptic system

$$L(x,D)u := \sum_{|\alpha|=2b} A_\alpha(x) D^\alpha u(x) = f(x),$$

where the principal coefficients A_α are assumed to be functions with vanishing mean oscillation *(VMO)*. In [9 - 11] A Calderón-Zygmund type theory in the framework of the classical Morrey spaces $L_{p\lambda}$ has been developed for linear and quasi-linear elliptic and parabolic systems with *VMO* principal coefficients. On the other hand, in the recent years an exhaustive Calderón-Zygmund theory has been elaborated both for elliptic and parabolic equations/systems in divergence form with *VMO* - coefficients in the framework of the various types of Morrey spaces (10, 11, 12 - 20]). Last generalizations of the spaces allows finer control on the local oscillation properties of a function near its singular points and that is why regularity results in $M_{p,\varphi}$ of solutions to PDEs with discontinuous coefficients are of great importance in the applications to differential geometry, stochastic control, nonlinear optimization, adaptive discontinuous Galerkin FEMs, etc.

In this paper we are going to extend the results obtained in [5, 17] to uniformly elliptic systems with discontinuous coefficients in the framework of generalized local Morrey spaces $M_{p,\varphi}^{\{x_0\}}$. The methods of proof used here are closer to that in the paper of Palagachev and Softova [19].

2. Preliminaries

For $x\in\mathbb{R}^n$ and $r>0$, let $B(x,r)$ denote the open ball centered at x of radius r. Let $\Omega\subset\mathbb{R}^n$, $n\geq3$ be a bounded domain, we denote by $|\Omega|$ the Lebesgue measure of Ω and $\Omega(x,r)=\Omega\cap B(x,r)$. $S^{n-1}=\{\xi\in\mathbb{R}^n: |\xi|=1\}$ is the unit sphere in $\mathbb{R}^n$; $M^{m\times m}$ is the set of $m\times m$-matrices.

In the following we give the definitions of the generalized local Morrey spaces $M_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ and generalized Morrey spaces $M_{p\varphi}(\mathbb{R}^n)$ and their weak versions, respectively.

Definition 2.1. ([5, 21) Let $x_0\in\Omega$, $1\leq p<\infty$ and φ be a positive measurable function on $\Omega\times(0,\infty)$. We denote by $M_{p,\varphi}^{\{x_0\}}(\Omega)$ and $M_{p\varphi}(\Omega)$ the generalized local Morrey space, the generalized Morrey space respectively, the spaces of all functions $f\in L_p(\Omega)$ with finite norms

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$$\|f\|_{M_{p,\varphi}^{\{x_0\}}}=\sup_{r>0}\varphi(x_0,r)^{-1}\left|B(x_0,r)\right|^{-\frac{1}{p}}\|f\|_{L_p(\Omega(x_0,r))}$$

and

$$\|f\|_{M_{p\varphi}(\Omega)}=\sup_{x\in\Omega,r>0}\varphi(x,r)^{-1}\left|B(x,r)\right|^{-\frac{1}{p}}\|f\|_{L_p(\Omega(x,r))},$$

respectively, where $L_p(\Omega(x, r))$ denotes the L_p - space of measurable functions f for which,

$$\|f\|_{L_p(\Omega(x,r))}=\left(\int_{\Omega(x,r)}\left|f(y)\right|^p dy\right)^{\frac{1}{p}}$$

Furthermore, by $WM_{p,\varphi}^{\{x_0\}}(\Omega)$ and $WM_{p\varphi}(\Omega)$ we denote the weak generalized Morrey space and weak generalized Morrey space respectively of all functions $f\in WL_p(\Omega)$ for which,

$$\|f\|_{WM_{p,\varphi}^{\{x_0\}}}=\sup_{r>0}\varphi(x_0,r)^{-1}\left|B(x_0,r)\right|^{-\frac{1}{p}}\|f\|_{WL_p(\Omega(x_0,r))}<\infty$$

and

$$\|f\|_{WM_{p,\varphi}}=\sup_{x\in\Omega,r>0}\varphi(x,r)^{-1}\left|B(x,r)\right|^{-\frac{1}{p}}\|f\|_{WL_p(\Omega(x,r))}<\infty$$

respectively, where $WL_p(\Omega)(x, r)$ denotes the weak L_p - space of measurable functions f for which,

$$\|f\|_{WL_p(\Omega(x,r))}=\sup_{t>0}t\cdot\left|\left\{y\in\Omega(x,r):\left|f(y)\right|>t\right\}\right|^{1/p}$$

For $\mathbf{u}=(u_1,...,u_m):\Omega\to\mathbb{R}^m$ we write $\left|\mathbf{u}\right|^2=\sum_{j\le m}\left|u_j\right|^2$. For any function f and any domain D with $f:\Omega\to\mathbb{R}^m$ we write

$$f_D=\frac{1}{|D|}\int_D f(y)dy,\ \ \|f\|_{p,D}^p=\|f\|_{L_p(D)}^p=\int_D\left|f(y)^p dy\right|$$

Definition 2.2. ([19]) The generalized local Sobolev-Morrey space $W_{p,\varphi}^{2b,\{x_0\}}(\Omega)$ and generalized Sobolev-Morrey space $W_{p,\varphi}^{2b}(\Omega)$ consists of all functions $u\in L_p(\Omega)$ with generalized derivatives $D^\alpha u$, $|\alpha|\le 2b$, belonging to $M_{p,\varphi}^{\{x_0\}}(\Omega)$ and $M_{p,\varphi}(\Omega)$, and endowed with the norm

$$\|u\|_{W_{p,\varphi}^{2b,\{x_0\}}(\Omega)}=\sum_{s=0}^{2b}\sum_{|\alpha|=s}\|D^\alpha u\|_{M_{p,\varphi}^{\{x,\varphi\}}(\Omega)}$$

and

$$\|u\|_{W_{p,\varphi}^{2b}(\Omega)}=\sum_{s=0}^{2b}\sum_{|\alpha|=s}\|D^\alpha u\|_{M_{p,\varphi}(\Omega)},$$

respectively.

For $\boldsymbol{u}\in L_p(\Omega;\mathbb{R}^m)$ we write $\|\boldsymbol{u}\|_{p,\Omega}$ instead of $\|\boldsymbol{u}\|_{L_p(\Omega;\mathbb{R}^m)}$. Analoqously, $\boldsymbol{u}=(u_1,...,u_m)\in W_{p,\varphi}^{2b,\{x_0\}}(\Omega;\ \mathbb{R}^m)$ and $\boldsymbol{u}=(u_1,...,u_m)\in W_{p,\varphi}^{2b}(\Omega;\ \mathbb{R}^m)$ means $u_k\in W_{p,\varphi}^{2b,\{x_0\}}(\Omega)$ and $u_k\in W_{p,\varphi}^{2b}(\Omega)$, and the norm $\|\boldsymbol{u}_k\|_{W_{p,\varphi}^{2b,\{x_0\}}(\Omega,\mathbb{R}^m)}$ and $\|\boldsymbol{u}\|_{W_{p,\varphi}^{2b}(\Omega,\mathbb{R}^m)}$ is given by $\sum_{k=1}^m\|u_k\|_{W_{p,\varphi}^{2b,\{x_0\}}(\Omega)}$ and $\sum_{k=1}^m\|u_k\|_{W_{p,\varphi}^{2b}(\Omega)}$, respectively.

We will use also a localized version $W_{p,\varphi,\mathrm{loc}}^{2b,\{x_0\}}(\Omega;\mathbb{R}^m)$ and $W_{p,\varphi,\mathrm{loc}}^{2b}(\Omega;\mathbb{R}^m)$ of $W_{p,\varphi}^{2b,\{x_0\}}(\Omega;\mathbb{R}^m)$ and $W_{p,\varphi}^{2b}(\Omega;\mathbb{R}^m)$ respectively, consisting of all functions $\mathbf{u}$ that belong to $\mathbf{u}\in W_{p,\varphi}^{2b,\{x_0\}}(\Omega';\mathbb{R}^m)$ and $\mathbf{u}\in W_{p,\varphi}^{2b}(\Omega';\mathbb{R}^m)$ for each $\Omega'\subset\Omega$.

Remark 2.1. (1) If $\varphi(x,r)=r^{\frac{\lambda-n}{p}}$ with $0<\lambda<n$, then $M_{p,\varphi}^{\{x_0\}}(\Omega)=L_{p,\lambda}^{\{x_0\}}(\Omega)$ is the local Morrey space, $M_{p\varphi}(\Omega)=L_{p,\lambda}(\Omega)$ is the classical Morrey space and $WM_{p,\varphi}^{\{x_0\}}(\Omega)=WL_{p,\lambda}^{\{x_0\}}(\Omega)$ is the weak Morrey space, $WM_{p,\varphi}(\Omega)=WL_{p,\lambda}(\Omega)$ is the weak Morrey space.

(2) If $\varphi(x,r)\equiv\left|B(x,r)\right|^{-\frac{1}{p}}$, then $M_{p,\varphi}^{\{x_0\}}(\Omega)=M_{p,\varphi}(\Omega)=L_p(\mathbb{R}^n)$ is the Lebesgue space and $WM_{p,\varphi}^{\{x_0\}}(\Omega)=WM_{p,\varphi}(\Omega)=WL_p(\Omega)$ is the weak Lebesgue space.

(3) It is clear that if $\varphi(x,r)=r^{\frac{\lambda-n}{p}}$ with $0<\lambda<n$, then $M_{p,\varphi}$ gives rise to the classical Morrey space $L_{p,\lambda}$, while $L_{p,1}\equiv L_p$ and $W_{p,1}^{2b}$ reduces to the classical parabolic Sobolev space W_p^{2b} (cf. [11]) when $\varphi\equiv 1$.

Definition 2.3. Let $K(x;\xi):\mathbb{R}^n\times(\mathbb{R}_n\backslash\{0\})\to\mathbb{R}$ be a variable Calderón-Zygmund kernel, i.e.,

1. for each fixed is $x\in\mathbb{R}^n$, $K(x;\cdot)$ is *a* Calderón-Zygmund kernel:
   a) $K(x;\cdot)\in C^\infty(\mathbb{R}_n\backslash\{0\})$
   b) $K(x;\mu\xi)=\mu^{-n}K(x,\xi)\ \ \forall\mu>0$
   c) $\int_{S^{n-1}}K(x;\xi)d\sigma=0\ \ \int_{S^{n-1}}\left|K(x;\xi)\right|d\sigma_\xi<\infty$
2. for every multi-index $\beta:\sup_{\xi\in S^{n-1}}\left|D_\xi^\beta K(x;\xi)\right|\le C(\beta)$ independently of x, where S^{n-1} is the unit sphere in $\mathbb{R}^n$.

Given a function $f\in L_1(\Omega)$, define the singular integral operator

$$Tf(x):=P.V.\int_{\mathbb{R}^n}K(x;x-y)f(y)dy$$

and its commutator with multiplication by a function $a\in L_\infty(\mathbb{R}^n)$ as

$$C[a,f](x):=P.V.\int_{\mathbb{R}^n}K(x;x-y)\left[a(y)-a(x)\right]f(y)dy=$$
$$=T(af)(x)-a(x)Tf(x)$$

In [22] Chiarenza and Frasca show boundedness of the Hardy-Littlewood maximal operator in $L_{p,\lambda}(\mathbb{R}^n)$ that allows them to prove continuity of fractional and classical Calderón-Zygmund operators in these spaces. Recall that integral operators of that kind appear in the representation formulae of the solutions of elliptic/parabolic equations and systems. Thus the continuity of the Calderón-Zygmund integrals implies regularity of the solutions in the corresponding spaces. In [3] Mizuhara studied the continuity in $M_{p,\varphi}(\mathbb{R}^n)$ of some classical integral operators. Nakai later extends the results of Chiarenza and Frasca in $M_{p,\varphi}(\mathbb{R}^n)$ by imposing certain integral and doubling conditions on φ (see [4]). Taking a weight $w(x, r)=\varphi(x, r)r^n$ the conditions of Mizuhara - Nakai become

$$\int_r^\infty\frac{\varphi(x,s)}{s}ds\le C\varphi(x,r),\ C^{-1}\frac{\varphi(x,t)}{(x,r)}\le C,\ \forall 0<r\le t\le 2r \tag{1}$$

where the constants do not depend on t, r and $x\in\mathbb{R}^n$.

In a series of papers, the first author studied the continuity in generalized Morrey spaces of sublinear operators generated by various integral operators, such as Calderón-Zygmund, Riesz, etc. (see [2, 5]). The following theorem, obtained in [2], extends Nakai's results in Morrey-type spaces with weight $w(x, r)=\varphi(x, r)r^n$.

Proposition 2.1. Let $1\le p<\infty$ and (φ_1, φ_2) satisfy the condition

$$\int_r^\infty\frac{\varphi_1(x,t)}{t}ds\le C\varphi_2(x,r)$$

where C does not depend on x and r. Then the Calderón-Zygmund integral operators T are bounded from $M_{p,\varphi_1}(\mathbb{R}^n)$ to $M_{p,\varphi_2}(\mathbb{R}^n)$ for $p>1$ and from $M_{1,\varphi_1}(\mathbb{R}^n)$ to the weak space $WM_{1,\varphi_2}(\mathbb{R}^n)$.

Later this results is extended on spaces with weaker condition on the weight pair (φ_1, φ_2) (see [23]). For more recent results on boundedness and continuity of singular integral operators in generalized Morrey and new function spaces and their applications in the differential equations theory, see [10, 11, 17-19, 21] and the references therein.

Proposition 2.2. Let $1<p<\infty$, $a\in$ BMO, φ be a weight satisfying: there exist positive constant $C>0$ such that

$$\int_r^\infty \left(1+In\frac{t}{r}\right)\frac{\varphi(x,s)}{s}ds \le C\varphi(x,r),\quad C^{-1}\frac{\varphi(x,t)}{\varphi(x,r)}\le C, \qquad (2)$$
$$\forall 0<r\le t\le 2r$$

where the constants do not depend on t, r and $x\in\Omega$.

Then there exists a positive constan $C=C(p,\varphi,K)$ such that

$$\|Tf\|_{Mp,\varphi(\Omega)} \le C\|f\|_{Mp,\varphi(\Omega)}$$

and

$$\|C[a,f]\|_{M_{p,\varphi}(\Omega)} \le C\|a\|_*\|f\|_{M_{p,\varphi}(\Omega)}.$$

In addition, if $a\in$VMO, then for each $\varepsilon>0$ there exists $r_0=r_0(\varepsilon,\gamma_a)>0$ such that for any $r\in(0,r_0)$ and any ball B_r the following inequality holds:

$$\|C[a,f]\|M_{p,\varphi}(B_r)\le C\in\|f\|_{M_{p,\varphi}(\Omega)}.$$

The L_p and $M_{p,\varphi}$ - boundedness of the operators T and C have been obtained in [16, 24] and [2, 5, 17], respectively. For the sake of completeness, we summarize these results here.

Proposition 2.3. Let $x_0\in\Omega$, $1<p<\infty$, $a\in$ BMO, φ be a weight satisfying

$$\int_r^\infty \left(1+In\frac{t}{r}\right)\frac{\operatorname*{ess\,inf}_{t<s<\infty}\varphi(x_0,s)s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}dt\le C\varphi(x_0,r) \qquad (3)$$

where C does not depend on r. Then there exists a positive constant $C=C(p,\varphi,K)$ such that

$$\|Tf\|_{M_{p,\varphi}^{\{x_0\}}(\Omega)}\le C\|f\|_{M_{p,\varphi}^{\{x_0\}}(\Omega)}$$

and

$$\|C[a,f]\|_{M_{p,\varphi}^{\{x_0\}}(\Omega)}\le C\|a\|\cdot\|f\|_{M_{p,\varphi}^{\{x_0\}}(\Omega)}.$$

In addition, if $a\in$VMO, then for each $\varepsilon>0$ there exists $r_0=r_0(\varepsilon,\gamma_a)>0$ such that for any $r\in(0,r_0)$ and any ball B_r the following inequality holds:

$$\|C[a,f]\|_{M_{p,\varphi}^{\{x_0\}}(B_r)}\le C\varepsilon\|f\|_{M_{p,\varphi}^{\{x_0\}}(\Omega)}.$$

Corollary 2.1. Let $1<p<\infty$, $a\in$ BMO, φ be a weight satisfying

$$\int_r^\infty \left(1+In\frac{t}{r}\right)\frac{\operatorname*{ess\,inf}_{t<s<\infty}\varphi(x,s)s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}dt\le C\varphi(x,r) \qquad (4)$$

where C does not depend on x and r. Then there exists a positive constant $C=C(p,\varphi,K)$ such that

$$\|Tf\|_{M_{p,\varphi}(\Omega)}\le C\|f\|_{M_{p,\varphi}(\Omega)}$$

and

$$\|C[a,f]\|_{M_{p,\varphi}(\Omega)}\le C\|a\|\cdot\|f\|_{M_{p,\varphi}(\Omega)}$$

In addition, if $a\in$VMO, then for each $\varepsilon>0$ there exists $r_0=r_0(\varepsilon,\gamma_a)>0$ such that for any $r\in(0,r_0)$ and any ball B_r the following inequality holds:

$$\|C[a,f]\|_{M_{p,\varphi}(B_r)}\le C\varepsilon\|f\|_{M_{p,\varphi}(\Omega)}.$$

3. Main result

Definition 3.1. ([19]) For $a\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and any $R>0$ set

$$\gamma_a(R):=\sup_{B_r,\,r\le R}\frac{1}{|B_r|}\int_{B_r}|a(y)-a_{B_r}|dy$$

where B_r is any ball in $\mathbb{R}^n$. We say that $a\in BMO$, if $\|a\|_*=\sup_{R>0}\gamma_a(R)<\infty$ and $a\in VMO$ with VMO- modulus γ_a, if $a\in BMO$ and $\lim_{R\to 0}\gamma_a(R)=0$.

We are interested in operators with discontinuous coefficients a_α^{jk} be longing to the Sarason function class VMO. For a matrix-valued function $A\in M^{m\times m}$ with entries $a^{jk}\in VMO$ we define the VMO-modulus of A as

$$\gamma_A=\sum_{j,k=1}^m\gamma_{a^{jk}}.$$

After this $\mathbf{u}:\Omega\to\mathbb{R}^m$, $m\ge1$, stands for the unknown function, $\mathbf{f}=(f_1,\dots,f_m):\Omega\to\mathbb{R}^m$ is a given vector-valued function and the coefficient matrix $\mathbf{A}_\alpha(x)\in M^{m\times m}$ has entries $\{a_\alpha^{jk}\}_{j,k=1}^m$, $a_\alpha^{jk}:\Omega\to\mathbb{R}$, which are measurable functions. Fixed an integer $b\ge1$, we deal with the $2b$-order linear system system

$$L(x,D)\mathbf{u}:=\sum_{|\alpha|=2b}\mathbf{A}_\alpha(x)D^\alpha\mathbf{u}(x)=\mathbf{f}(x)\ \text{a.e.in}\ \Omega \qquad (5)$$

that is equivalent to the system of differential equations

$$\sum_{k=1}^m\sum_{|\alpha|=2b}a_\alpha^{jk}D^\alpha u_k=\sum_{k=1}^m l^{jk}(x,D)u_k=f^j(x), \qquad (6)$$
$$j=1,\dots,m.$$

The entries $l^{jk}(x,D)$ of the matrix differential operator $L(x,D)$ are homogeneous polynomials of degree $2b$, that is, (3.3)

$$l^{jk}(x,\xi):=\sum_{|\alpha|=2b}a_\alpha^{jk}(x)\xi^\alpha,\ \xi\in\mathbb{R}^n,\ \xi^\alpha:=\xi_1^{\alpha_1}\xi_2^{\alpha_2}\dots\xi_n^{\alpha}. \qquad (7)$$

The operator $L(x,D)$ is supposed to be uniformly elliptic that means the characteristic determinant of $L(x,\xi)$ is non-vanishing for a.e. $x\in\Omega$ and all $\xi\ne0$. Due to the homogeneity of l^{jk} this condition can be written as

$$\exists\delta>0:\ \det\left\{\sum_{|\alpha|=2b}\mathbf{A}_\alpha(x)\xi^\alpha\right\}\le\delta|\xi|^{2bm} \qquad (8)$$

for almost all $x\in\Omega$ and all $\xi\in\mathbb{R}^n$.

Fix the coefficients of (5) at $x_0\in\Omega$ and consider the constant coefficients operator

$$L(x_0,D):=\sum_{|\alpha=2b|}\mathbf{A}(x_0)D^\alpha.$$

Then the $2bm$-order differential operator

$$L(x_0,D):=\det L(x_0,D)=\det\left\{\sum_{|\alpha|=2b}a_\alpha^{jk}(x_0)D^\alpha\right\}_{j,k=1}^m \qquad (9)$$

is elliptic as it follows from (8), and let $\tilde\Gamma(x_0;x-y)$ be its fundamental solution. If the space dimension n is odd, then

$$\tilde\Gamma(x_0;x-y)=|x-y|^{2bm-n}P\left(x_0;\frac{x-y}{|x-y|}\right) \qquad (10)$$

with $P(x_0;\xi)$ being a real analytic function of $\xi\in S^{n-1}$. If n

is even, it is enough to introduce a fictitious new variable x_{n+1} and extend all functions as constants with respect to it (see [25]). Let $\{L_{jk}(x_0,\xi)\}_{j,k=1}^{m}$ be the cofactor matrix of $\{l^{jk}(x_0,\xi)\}_{j,k=1}^{m}$. Then $L_{jk}(x_0, D)$ are differential operators of up to $2b\,(m-1)$ or indentically zero. Since

$$\sum_{k=1}^{m} l^{ik}(x_0,\xi)L_{jk}(x_0,\xi)=\delta_{ij}L(x_0,\xi) \tag{11}$$

with the Kronecker symbol δ_{ij}, the fundamental matrix of $L(x_0, D)$ is given by

$$\Gamma(x_0;x)=\left\{\Gamma^{jk}(x_0;x)\right\}_{j,k=1}^{m}=\left\{L_{kj}(x_0,D)\tilde{\Gamma}(x_0;x)\right\}_{j}^{m_{k=1}}.$$

Let $B_r\subset\Omega$ be such that $x_0\in B_r$, $\mathbf{v}\in C_0^\infty(B_r)$ and let us write

$$L(x_0, D)\mathbf{v}(x)=(L(x_0, D)-L(x, D))\mathbf{v}(x)+L(x, D)\mathbf{v}(x).$$

Using the standard approach [15, 16, 25] we obtain an explicit representation formula for $\mathbf{v}$ via *Newtonian potentials*

$$\begin{aligned}\mathbf{v}(x)&=\int_{B_r}\Gamma(x_0;x-y)L\mathbf{v}(y)dy+\\&+\int_{B_r}\Gamma(x_0;x-y)\big(L(x_0,D)-L(y,D)\big)\mathbf{v}(y)dy.\end{aligned} \tag{12}$$

Taking the α-derivatives with $|\alpha|=2b$ and then unfreezing the coefficients putting $x_0=x$ we get

$$\begin{aligned}D^\alpha\mathbf{v}(x)&=p.v.\int_{B_r}D^\alpha\Gamma(x;x-y)L\mathbf{v}(y)dy+\\&+\sum_{|\alpha|=2b}p.v.\int_{B_r}D^\alpha\Gamma(x;x-y)\big(\mathbf{A}_{\alpha'}(x)-\mathbf{A}_{\alpha'}(y)\big)D^{\alpha'}\mathbf{v}(y)dy+\\&+\int_{S^{n-1}}D^{\beta^s}\Gamma(x;y)\nu_s d\sigma_y L\mathbf{v}(x)=:T_\alpha(L\mathbf{v})(x)+\\&+\sum_{|\alpha|=2b}C_\alpha\left[\mathbf{A}_{\alpha'},D^{\alpha'}\mathbf{v}(x)\right]+L\mathbf{v}(x)Q_\beta|(x)|\end{aligned} \tag{13}$$

where the derivatives $D^\alpha\Gamma(\cdot;\cdot)$ and $D^{\beta s}\Gamma(\cdot;\cdot)$ are taken with respect to the second variable, the multi-indices β^s are given by

$$\beta^s=(\alpha_1,\ldots,\alpha_{s-1},\alpha_s-1,\alpha_{s+1},\ldots,\alpha_n),\ |\beta^s|=2b-1$$

and $\nu=(\nu_1,\ldots,\nu_n)$ is the outer normal to S^{n-1}. Let us note that T_α are Calderón-Zygmund type singular integral operators, C_α are commutators of T_α with *VMO* functions, and Q_β are bounded integrals (cf. [15, 16, 10]).

The proof of our main result is based on some real analysis results on the boundedness of Calderón-Zygmund type singular integral operators and their commutators obtained in [2, 5, 17]. In order to estimate the seminorms Θ_s we need the following interpolation inequality which follows from [17, 10].

Lemma 3.1. There is a constant C, independent of r, such that

$$\Theta_s\le\varepsilon\Theta_{2b}+\frac{C}{\varepsilon^{s/(2b-s)}}\Theta_0\quad\text{for each }\varepsilon\in(0,2), \tag{14}$$

where $\Theta_s=\sup\limits_{0<\theta<1}\left[\theta(1-\theta)r\right]^s\|D^s\boldsymbol{u}\|_{M_{p,\varphi}^{\{x_0\}}(B_{\theta r})}$.

Proof. Let $\theta_0\in(0,1)$ be such that

$$\Theta_s\le 2\left[\theta_0(1-\theta_0)r\right]^s\|D\mathbf{u}\|^s_{M_{p,\varphi}^{\{x_0\}}(B_{\Theta_0 r})}.$$

By interpolation and scaling arguments we obtain

$$\|D^s\mathbf{u}\|_{M_{p,\varphi}^{\{x_0\}}(B_{\theta_0 r})}\le\delta^{2b-s}\|D^{2b}\mathbf{u}\|_{M_{p,\varphi}^{\{x_0\}}(B_{\theta_0 r})}+\frac{C'}{\delta^s}\|\mathbf{u}\|_{M_{p,\varphi}^{\{x_0\}}(B_{\Theta_0 r})}$$

for and hence

$$\begin{aligned}\Theta_s&\le 2\left[\theta_0(1-\theta_0)r\right]^s\delta^{2b-s}\|D^{2b}\mathbf{u}\|_{M_{p,\varphi}^{\{x_0\}}(B_{\Theta_0 r})}+\\&+\frac{2C'\left[\theta_0(1-\theta_0)r\right]^s}{\delta^s}\|\mathbf{u}\|_{M_{p,\varphi}^{\{x_0\}}(B_{\Theta_0 r})}.\end{aligned}$$

The assertion follows choosing

$$\delta=(\varepsilon/2)^{\frac{1}{2b-s}}\left[\theta_0(1-\theta_0)r\right]<\theta_0 r\ \text{ for any }\varepsilon\in(0,2).$$

We give our main result in the following.

Theorem 3.1. Assume that (8) is provided and $\boldsymbol{A}_\alpha=\{a_\alpha^{jk}\}\in VMO(\Omega)\cap L^\infty(\Omega)$. Let $x_0\in\Omega$, $\boldsymbol{u}\in W_{p,\mathrm{loc}}^{2b,\{x_0\}}(\Omega;\mathbb{R}^m)$ be a strong solution to (5) with $p\in(1,\infty)$. Let $\boldsymbol{f}\in M_{p,\varphi}^{\{x_0\}}(\Omega;\mathbb{R}^m)$ such that w is satisfy (3). Then $\boldsymbol{u}\in W_{p,\varphi,\mathrm{loc}}^{2b,\{x_0\}}(\Omega;\mathbb{R}^m)$ and

$$\|\boldsymbol{u}\|_{W_{p,\varphi}^{2b,\{x_0\}}(\Omega',\mathbb{R}^m)}\le C\left(\|\boldsymbol{f}\|_{M_{p,\varphi}^{\{x_0\}}(\Omega)}+\|\boldsymbol{u}\|_{M_{p,\varphi}^{\{x_0\}}(\Omega'')}\right) \tag{15}$$

for all $\Omega'\subset\Omega''\subset\Omega$, where the constant C depends on n, p, m, b, ω, $\|A_\alpha\|_{\infty;\Omega}$, the *VMO*-moduli γ_{A_α} of the coefficients and on dist $(\Omega',\partial\Omega'')$.

Proof. Fix an arbitrary $x_0\in\operatorname{supp}u$ and let $B_r\equiv B(x_0,r)\Subset\Omega$. Consider $\mathbf{v}\in W_{0,p}^{2b,\{x_0\}}(B(x_0,r))$ (the closure of $C_0^\infty(B(x_0,r))$ with respect to the norm in $W_p^{2b,\{x_0\}}B(x_0,r)$) with $\operatorname{supp}v\subset B(x_0, r)$. Then (13), Proposition 2.1 and $A_\alpha\in VMO(\Omega)$ imply that for each $\varepsilon>0$ there exists $r_0=r_0(\varepsilon,\gamma_{A_\alpha})$ such that

$$\|D^{2b}\mathbf{v}\|_{M_{p,\varphi}^{\{x_0\}}(B_r)}\le C\left(\|L\mathbf{v}\|_{M_{p,\varphi}^{\{x_0\}}(B_r)}+\varepsilon\|D^{2b}\mathbf{v}\|_{M_{p,\varphi}^{\{x_0\}}(B_r)}\right)$$

when ever $r<r_0$. Choosing ε small enough we obtain

$$\|D^{2b}\mathbf{v}\|_{M_{p,\varphi}^{\{x_0\}}(B_r)}\le C\|L\mathbf{v}\|_{M_{p,\varphi}^{\{x_0\}}(B_r)}. \tag{16}$$

Let $\theta\in(0,1)$, $\theta'=\theta(3-\theta)/2>0$ and define the cut-off function $\varphi(x)\in C_0^\infty(B_r)$ such that

$$\varphi(x)=\begin{cases}1 & x\in B_{\theta r}(x_0)\\ 0 & x\notin B_{\theta' r}(x_0)\end{cases}.$$

Since $\theta'-\theta=\theta(1-\theta)/2$, direct calculations give

$$|D^s\varphi|\le C(s)\left[\theta(1-\theta)r\right]^{-s},\quad\forall s=1,2,\ldots,2b.$$

Setting $\mathbf{v}=\varphi\mathbf{u}$ in (16) we obtain

$$\begin{aligned}\|D^{2b}\mathbf{u}\|_{M_{p,\varphi}^{\{x_0\}}(B_{\theta r})}&\le\|D^{2b}\mathbf{v}\|_{M_{p,\varphi}^{\{x_0\}}(B_{\theta' r})}\le C\|L\mathbf{v}\|_{M_{p,\varphi}^{\{x_0\}}(B_{\theta' r})}\\&\le C\|\mathbf{f}\|_{M_{p,\varphi}^{\{x_0\}}(B_{\theta' r})}+\sum_{s=1}^{2b-1}\frac{\|D^{2b-s}\mathbf{u}\|_{M_{p,\varphi}^{\{x_0\}}(B_{\theta' r})}}{\left[\theta(1-\theta)r\right]^s}+\frac{\|\mathbf{u}\|_{M_{p,\varphi}^{\{x_0\}}(B_{\theta' r})}}{\left[\theta(1-\theta)r\right]^{2b}}.\end{aligned}$$

Because of the choice of θ' we have $\theta(1-\theta)\le2\theta'(1-\theta')$ that implies

$$\begin{aligned}\left[\theta(1-\theta)r\right]^{2b}\|D^{2b}\mathbf{u}\|_{M_{p,\varphi}^{\{x_0\}}(B_{\theta r})}&\le C\Big(\left[\theta'(1-\theta')r\right]^{2b}\|\mathbf{f}\|_{M_{p,\varphi}^{\{x_0\}}(B_{\theta' r})}+\\&+\sum_{s=1}^{2b-1}\left[\theta'(1-\theta')r\right]^s\|D^s\mathbf{u}\|_{M_{p,\varphi}^{\{x_0\}}(B_{\theta' r})}+\|\mathbf{u}\|_{M_{p,\varphi}^{\{x_0\}}(B_{\theta' r})}\Big).\end{aligned} \tag{17}$$

Setting $\Theta_s=\sup\limits_{0<\theta<1}\left[\theta(1-\theta)r\right]^s\|D^s\mathbf{u}\|_{M_{p,\varphi}^{\{x_0\}}(B_{\theta r})}$ we can rewrite (17) as

$$\Theta_{2b}\le C\left(r^{2b}\|\mathbf{f}\|_{M_{p,\varphi}^{\{x_0\}}(B_r)}+\sum_{s=1}^{2b-1}\Theta_s+\Theta_0\right). \tag{18}$$

Choosing suitable $\varepsilon\in(0,2)$, and applying (14) we get

$$\Theta_{2b}\le C\left(r^{2b}\|\mathbf{f}\|_{M_{p,\varphi}^{\{x_0\}}(B_r)}+\Theta_0\right).$$

In Lemma 3.1 fixing $\theta=1/2$ at the seminorm Θ_s we obtain the following Caccioppoli-type estimate:

$$\|D^{2b}\mathbf{u}\|_{M_{p,\varphi}^{\{x_0\}}(B_{r/2})}\le C\left(\|\mathbf{f}\|_{M_{p,\varphi}^{\{x_0\}}(\Omega)}+r^{-2b}\|\mathbf{u}\|_{M_{p,\varphi}^{\{x_0\}}(B_r)}\right). \tag{19}$$

The desired estimate (15) follows now by means of standard covering arguments with balls $B_{r/2}$ for r<dist $(\Omega',\partial\Omega'')$ and partition of unity over Ω′ subordinated to this covering.

Corollary 3.1. Assume that (8) is provided and $\boldsymbol{A}_\alpha=\{a_\alpha^{jk}\}\in VMO(\Omega)\bigcap L^\infty(\Omega)$. Let $\boldsymbol{u}\in W^{2b,}_{p,\varphi,\mathrm{loc}}(\Omega;\mathbb{R}^m)$ be a strong solution to (5) with $p\in(1,\infty)$. Let $\boldsymbol{f}\in M_{p,\varphi}(\Omega;\mathbb{R}^m)$ such that *w* is satisfy (4). Then $\boldsymbol{u}\in W^{2b,}_{p,\varphi,\mathrm{loc}}(\Omega;\mathbb{R}^m)$ and

$$\|\boldsymbol{u}\|_{W^{2b}_{p,\varphi}(\Omega',\mathbb{R}^m)}\le C\left(\|\boldsymbol{f}\|_{M_{p,\varphi}(\Omega)}+\|\boldsymbol{u}\|_{M_{p,\varphi}(\Omega'')}\right)$$

for all $\Omega'\subset\Omega''\subset\Omega$, where the constant *C* depends on *n, p, m, b, ω,* $\|\boldsymbol{A}_\alpha\|_{\infty;\Omega}$, the *VMO*-moduli γ_{A_α} of the coefficients and on dist $(\Omega',\partial\Omega'')$.

From Corollary 3.1 it is easy to obtain the following result.

Corollary 3.2. [19] Assume that (8) is provided and $\boldsymbol{A}_\alpha=\{a_\alpha^{jk}\}\in VMO(\Omega)\bigcap L^\infty(\Omega)$. Let $\boldsymbol{u}\in W^{2b,}_{p,\varphi,\mathrm{loc}}(\Omega;\mathbb{R}^m)$ be a strong solution to (5) with $p\in(1,\infty)$. Let $\boldsymbol{f}\in M_{p,\varphi}(\Omega;\mathbb{R}^m)$ such that *w* is satisfy (2). Then $\boldsymbol{u}\in W^{2b,}_{p,\varphi,\mathrm{loc}}(\Omega;\mathbb{R}^m)$ and

$$\|\boldsymbol{u}\|_{W^{2b}_{p,\varphi}(\Omega',\mathbb{R}^m)}\le C\left(\|\boldsymbol{f}\|_{M_{p,\varphi}(\Omega)}+\|\boldsymbol{u}\|_{M_{p,\varphi}(\Omega'')}\right)$$

for all $\Omega'\subset\Omega''\subset\Omega$, where the constant *C* depends on *n, p, m, b, ω,* $\|\boldsymbol{A}_\alpha\|_{\infty;\Omega}$, the *VMO*-moduli γ_{A_α} of the coefficients and on dist $(\Omega',\partial\Omega'')$.

Remark 3.1. Corollary 3.2 was proved in [19]. Note that condition (4) in Corollary 2.1 is weaker than condition (2) in Proposition 2.2. Indeed, if condition (2) holds, then

$$\int_r^\infty\left(1+\mathrm{In}\frac{t}{r}\right)\frac{\operatorname*{ess\,inf}_{t<s<\infty}\varphi(x,s)s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}dt\le\int_r^\infty\left(1+\mathrm{In}\frac{t}{r}\right)\varphi(x,t)\frac{dt}{t},$$

$$r\in(0,\infty)$$

so condition (4) holds.

On the other hand the function

$$\varphi(x,r)=r^{\beta-\frac{n}{p}}\left|\sin\left(\max\left\{1,\frac{\pi}{r}\right\}\right)\right|,\ 0<\beta<\frac{n}{p}$$

satisfy condition (4) in the caser $r\in(0,1)$, $\operatorname*{ess\,inf}_{r<s<\infty}\varphi(x,s)s^{\frac{n}{p}}=0$ and,

$$\int_r^\infty\left(1+\mathrm{In}\frac{t}{r}\right)\frac{\operatorname*{ess\,inf}_{r<s<\infty}\varphi(x,s)s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}dt\approx\begin{cases}0 & \text{if } r\in(0,1)\\ r^{\beta-\frac{n}{p}} & \text{if } r\in(1,\infty)\end{cases}$$

$$\lesssim\varphi(x,r),\ r\in(0,\infty)$$

but do not satisfy condition (2).

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