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SOLUTION OF THE DIRECT AND INVERSE PROBLEM OF GAS LIFT OIL PRODUCTION PROCESS BY THE OPTIMAL CONTROL METHOD

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ABSTRACT

The article presents a numerical method for solving the direct and inverse problems of the gas-lift process of oil production described by one-dimensional Navier-Stokes equations for compressible gas. To solve the direct problem, a family of difference schemes was developed and an analysis of the correctness of the difference problem was carried out depending on the parameter. The solution of the inverse problem is reduced to an optimal control problem, where the target functional is formed using an additional condition. To minimize the target functional, the gradient method is used, and its gradient is calculated by solving the conjugate problem. The conjugate problem contains important information about the solution of the direct problem and is based on the Lagrange identity and the condition of equality to zero of the integral terms. A feature of the conjugate problem is its retrospective nature, since the additional condition on the volumetric gas flow rate and pressure is specified at a certain point in time. The iteration method determines the initial conditions for the volumetric gas flow rate and pressure through the solution of the conjugate retrospective problem. The conducted computational experiment confirmed that the proposed algorithm can be used to determine the initial values of the volumetric gas flow rate and pressure with high accuracy under a given additional condition. The developed method also allows plotting the performance curve of the gas lift process.

Keywords: gas-lift oil production process; Navier-Stokes equations; conjugate equation; inverse problem; optimal control; gradient method; finite-difference method.

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Introduction

Gas lift is an oil production method [1-8] in which gas is injected into the annulus of a well to reduce the density of the gas-liquid mixture (GLM) and facilitate its rise to the surface. This process occurs in two main areas:

- Annulus: the space between the well wall and the tubing where the gas moves.
- Lift: the inner tube through which the oil, gas and water mixture rises to the wellhead.

The model proposed in [9] describes the movement of gas and gas-liquid mixture in these zones using a system of hyperbolic equations with partial derivatives [10]. These equations take into account the pressure P and the volumetric flow rate of the injected gas Q as the main parameters influencing the transportation of the mixture.

This [11] paper presents a mathematical method for optimizing oil production from wells using a gas lift system. The objective of the paper is to determine the cost-effective level of oil production and reduce production costs by minimizing the consumption of gas used to lift the oil [12-16]. To do this, well performance data was collected and used in the PIPESIM modeling application. Based on this data, performance curves for each well were constructed. Next, a nonlinear multi-criteria programming model was developed to optimize oil production.

In the work [17] the study of the method of optimization and distribution of gas lift for increasing oil recovery by genetic algorithm is considered.

In works [18-20] a mathematical model based on Darcy's law is used. This is a very simple equation in the right part of which is the force of gravity, the force of friction and acceleration.

The pressure is replaced by the gas density according to the equation of state, and the density of the mixture is considered as a linear combination of the gas and liquid densities [21].

Later, the authors of the work [18] used genetic algorithms to solve the problem of maximizing oil production. The rate of liquid production from a production well is illus-

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trated by a combination of the inflow productivity (IPR) [22] and the vertical lift productivity (VLP).

In [23], the gas lift performance curve (GLPC) is constucted for the given wells as a result of the experiment. Using this curve, the effect of gas injection rate on the liquid flow rate can be estimated. This allows the gas injection rate to be determined to achieve the desired performance. Studying GLPC allows the optimum gas injection rate to be found.

In [24], GLPC was obtained from field data by measuring gas injection rate and liquid production rate. The measured data were interpolated to obtain GLPC. From field data, GLPC was constructed using the least squares method as a quadratic polynomial function.

In [25], a new function is proposed to improve the previous GLPC quadratic polynomial by adding a logarithmic term.

In the work of Sukarno P. et al. [26] an exponential function for fitting GLPC from field data is proposed. However, exponential GLPC only describes gas injection well.

Researchers S. Guet and G. Oams [19, 20, 27] used a piecewise linear function to fit GLPC to field data, which has good application prospects.

All these works used semi-empirical approaches to study the performance curve of the gas lift process.

Today, a common method for solving inverse problems of mathematical physics is to reduce them to optimal control problems [28-30]. One of the current problems of modern optimal control is the control of the behavior of objects, the change of which is described using partial differential equations [31, 32]. The goal of control is to transfer the studied object from one known state to another by influencing some of its parameters. Such tasks were first formulated in the works of J. L. Lions [33, 34].

The right side of an equation or system of equations can be used as a control function. The works of V. I. Agashkov [35] are devoted to solving such problems using the method of conjugate equations.

Many works consider boundary controls, i.e. control by means of boundary conditions. The works of V. I. Ilyin and E. I. Moiseev [36] are devoted to the study of boundary control problems for the equation of string oscillations, in which control functions were obtained in explicit form that transfer the string from a given initial state to a given final state in a certain time. In this case, various types of boundary controls were considered.

In the work of G. I. Marchuk [37] the concept of conjugate operators and equations is given and their possible applications in mathematical modeling and computational mathematics are noted. The properties of conjugate operators have been studied quite fully for linear operators in Hilbert and Banach spaces and are reflected in many monographs.

In the work [38] a method of fictitious regions with the idea of conjugate optimization is proposed, which allows constructing a homogeneous difference scheme [39, 40] in the entire extended region. In this case, a reasonable continuation of the coefficients of the main equation leads to the convergence of the solution of the problem in the original region to the desired solution, which is confirmed by mathematically proven statements and the results of numerical calculations. To minimize the Lagrange functional, a conjugate gradient method was used, which allows finding an effective optimal solution by iterative refinement. In this case, it is necessary to calculate the gradient of the Lagrange functional, which leads to the formulation of the conjugate problem. The formulation of the conjugate problem is given, and the calculation of the gradient of the functional, which depends on the solution of the conjugate problem, is described. Based on the numerical results of the work, conclusions are made that the use of the gradient method, the conjugate problem and the method of fictitious regions are an effective approach for solving complex optimization problems with constraints. In this paper, the method is first developed for the Burgers equation. An auxiliary and conjugate problem for the Burgers equation is formulated. An iterative algorithm for an approximate solution of the auxiliary problem is developed. An estimate of the conditional stability of the conjugate problem is obtained by the method of energy inequalities. A theorem on the estimate of conditional stability is proved. The efficiency of using such a modification is shown on a model problem. This method is very convenient in terms of programming automation. Further, the proposed algorithm is developed to solve the Navier-Stokes equation.

In the article by A. V. Arguchintsev, V. P. Poplevko [41] the problem of optimal control of a system of semilinear hyperbolic equations is considered, in which the boundary conditions are determined from a system of ordinary differential equations with delay. The problem of modeling the dynamics of non-interacting populations is considered, taking into account the age distribution of individuals. The goal of the control problem may be to achieve specified population densities at the final time. For this problem, a non-classical necessary optimality condition is obtained, which is based on the use of a special control variation that ensures the smoothness of control functions. A method for improving admissible controls is proposed.

The work [42] considers the development of methods for solving optimal control problems [43] in the class of smooth control actions, taking into account such constraints on controls that are characteristic of inverse problems of mathematical physics.

The numerical implementation of the method is carried out for a system of first-order hyperbolic equations of the linearized theory of «shallow water». It is assumed that at the final moment of time the wave profile is known. The inverse problem is interpreted as a problem of minimizing a quadratic functional. Then the «shallow water» model is reduced to an invariant form. For the numerical solution, a difference scheme of the method of characteristics is used.

Inverse problems of various types are encountered in everyday life. The works [44, 45] are devoted to the study of the application of numerical methods to solving problems related to acoustic equations, with a special emphasis on problems of significant practical importance both in the field of medical imaging and in theoretical acoustics.

In the work of Temirbekov N. M. [46], the Bubnov-Galerkin projection method was used to solve the Fredholm integral equation of the first kind, where Legendre wavelets [47] were used as basis functions. In the Galerkin method, the expansion was carried out using these bases, which led to a system of linear algebraic equations for calculating the coefficients. The resulting system was solved by the conjugate gradient method.

The paper [48] examines methods for determining annular pressure during gas lift production. The authors analyze various approaches to calculating annular pressure at a given depth, which is critical for the efficient design of gas lift systems. Particular attention is paid to comparing theoretical models with field data, and practical recommendations for the application of these methods in real operating conditions are discussed.

[49] presents a closed-loop iterative gas lift well optimization workflow developed and implemented by ExxonMobil on over 1300 wells in the Perm Basin. Using machine learning and artificial intelligence techniques, the system automatically runs multiple tests at varying flow rates, analyzes the results, and remotely implements changes to the settings to optimize well performance.

In [50], the authors conduct a comparative analysis of high-pressure gas lift (HPGL) and electric submersible pumps (ESP) in the operation of unconventional wells in the Wolfcamp formation. The study covers operational, economic and production aspects of the application of both technologies.

The article [51] considers the application of robust model predictive control (MPC) to optimize gas distribution in gas lift systems. The work takes into account the uncertainties and limitations of production facilities, which makes the approach particularly relevant for real field conditions. Modifying the constraints within the MPC framework allows achieving stability and control efficiency in the presence of changing production conditions and resource constraints.

The article [52] considers the problems of optimal gas distribution between several wells in offshore platform conditions. A complex model of gas lift production is used, taking into account the interactions between wells and restrictions on the total gas lift resource. The authors apply numerical optimization methods and propose an algorithm that increases the efficiency of production. The work is relevant for integrated optimization problems and is an example of the application of modern computational approaches in real industrial conditions.

The article [53] proposes a methodology for determining the optimal paraffin removal interval in gas-lift wells. The work combines data analysis, mathematical modeling, and empirical criteria to predict paraffin accumulation and the moment of intervention. This topic has practical significance for maintaining stable well operation and improving production efficiency in fields with a high tendency to deposits.

The article [54] examines the main technological and management problems that arise during the operation of gas lift systems and analyzes existing approaches to their optimization. Particular attention is paid to methods for improving production efficiency, including the selection of the optimal gas injection mode and the integration of intelligent technologies. The work is a valuable review for researchers and engineers working in the field of automation and management of oil production.

In the work [55] the authors present the application of an adaptive gradient-free optimization method (Mesh Adaptive Direct Search, MADS) to solve the problem of optimal gas lift. The study is conducted under conditions of integrated modeling of fields, wells and ground facilities. This method allows taking into account the complex interaction of all elements of the production system and demonstrates high efficiency under conditions of uncertainty and nonlinearity of the problem. The work is of interest both from the point of view of numerical methods and practical implementation in the oil and gas industry.

1. Statement of the direct problem

The mathematical model of the operation of a gas lift well is described by the following system of Navier-Stokes equations for compressible gas [9]

$$\frac{\partial P}{\partial t} = -\frac{c^2}{\overline{F}} \cdot \frac{\partial Q}{\partial x}, \qquad t \ge 0, x \in \left(0; 2l\right), \tag{1}$$

$$\frac{\partial Q}{\partial t} = -\overline{F} \cdot \frac{\partial P}{\partial x} - 2a \cdot Q, \qquad t \ge 0, x \in \left(0; 2l\right), \tag{2}$$

where

$$c = \begin{cases} c_1, & x \in \left(0; l\right) \\ c_2, & x \in \left(l; 2l\right) \end{cases}, \quad \overline{F} = \begin{cases} \overline{F}_1, & x \in \left(0; l\right) \\ \overline{F}_2, & x \in \left(l; 2l\right) \end{cases},$$

$$a = \begin{cases} a_1, & x \in \left(0; l\right) \\ a_2, & x \in \left(l; 2l\right) \end{cases}$$

initial conditions

$$P\left(0, x\right) = P^0\left(x\right), \; Q\left(0, x\right) = Q^0\left(x\right) \tag{3}$$

and boundary conditions

$$P\left(t, 0\right) = P_0\left(t\right), \;\; Q\left(t, 0\right) = Q_0\left(t\right) \quad \text{at} \quad x = 0, \tag{4}$$

$$\begin{aligned} P\left(t, l+0\right) = P\left(t, l-0\right) + P_{res}\left(t\right), Q\left(t, l+0\right) = \\ = Q_0\left(t, l-0\right) + Q_{res}\left(t\right) \quad \text{at} \quad x = l, \end{aligned} \tag{5}$$

Here, is time, x is the coordinate along the well depth, P is the pressure, Q is the volumetric gas flow rate, $\overline{F}$ is the cross-sectional area of the well, c is the speed of sound in the liquid, a is the coefficient, Q_{res} is the volumetric gas flow rate in the reservoir, P_{res} is the reservoir pressure, $P^0(x)$ is the initial gas pressure distribution, $Q^0(x)$ is the initial volumetric flow rate of the injected gas, l is the well depth.

In the direct problem, it is necessary to find $P(t, x)$ and $Q(t, x)$ given the functions $P^0(x)$, $Q^0(x)$, $P_0(t)$, $Q_0(t)$, $P_{res}(t)$, $Q_{res}(t)$.

Equation (1) is the continuity equation, and (2) is the equation of gas motion.

2. Study of approximation and stability of a family of difference schemes

Let us consider an explicit difference scheme for problem (1)-(5)

$$\frac{P_i^{n+1} - P_i^n}{\tau} = -\frac{c^2}{\overline{F}} \cdot \frac{Q_i^n - Q_{i-1}^n}{h} \tag{6}$$

$$\frac{Q_i^{n+1} - Q_i^n}{\tau} = -\overline{F} \cdot \frac{P_i^n - P_{i-1}^n}{h} - 2a \cdot Q_i^{n+1} \tag{7}$$

initial conditions

$$P_i^0 = P^0\left(x_i\right), \;\; Q_i^0 = Q^0\left(x_i\right), \;\; i = \overline{0, \;\; N_x} \tag{8}$$

and boundary conditions respectively

$$P_0^j = P_0\left(t_j\right), \;\; Q_0^j = Q_0\left(t_j\right), \;\; i = 0, 1, \ldots, N_x, \tag{9}$$

$$P_{\frac{N_x}{2}}^j = P_{\frac{N_x}{2}-1}^j + P_{res}, \;\; Q_{\frac{N_x}{2}}^j = Q_{\frac{N_x}{2}-1}^j + Q_{res} \tag{10}$$

We will consider the implicit scheme for the equations in the form

$$P_i^{j+1} = P_i^j - \gamma_1 \cdot \left(Q_i^{j+1} - Q_{i-1}^{j+1}\right), \tag{11}$$

$$Q_i^{j+1} = \left[Q_i^j - \gamma_2 \cdot \left(P_i^{j+1} - P_{i-1}^{j+1}\right)\right] / \left(1 + 2a\tau\right), \tag{12}$$

where

$$\gamma_1 = \frac{\tau c^2}{\overline{F} \cdot h}, \;\; \gamma_1 = \frac{\tau \cdot \overline{F}}{h}$$

From this it is clear that the count can be started from the point i=1, j=0. Then

$$P_1^1 = P_1^0 - \gamma_1 \cdot \left(Q_1^1 - Q_0^1\right), \tag{13}$$

$$Q_1^1 = \left[Q_1^0 - \gamma_2 \cdot \left(P_1^1 - P_0^1\right)\right] / \left(1 + 2a\tau\right) \tag{14}$$

We multiply equation (14) by $-\gamma_1$ and sum it with the first one, we get

$$P_1^1 = P_1^0 + \gamma_1 \cdot Q_0^1 - \frac{\gamma_1}{1+2a\tau} \cdot Q_1^0 + \frac{\gamma_1\gamma_2}{1+2a\tau} \cdot P_1^1 - \frac{\gamma_1\gamma_2}{1+2a\tau} \cdot P_0^1$$

From here we find P_1^1

$$P_1^1 = \frac{1+2a\tau}{1+2a\tau - \gamma_1\gamma_2} \times$$
$$\times \left[P_1^0 + \gamma_1 \cdot Q_0^1 - \frac{\gamma_1}{1+2a\tau} \cdot Q_1^0 - \frac{\gamma_1\gamma_2}{1+2a\tau} \cdot P_0^1\right] \tag{15}$$

Substituting P_1^1 into (14) we determine Q_1^1.

Knowing P_1^1, Q_1^1, we can calculate all values of P_1^j, Q_1^j up to some j=j_0, then, setting i=2, find P_2^j, Q_2^j for 0≤j≤j_0, etc.

In general, to determine P_i^{j+1} we obtain

$$P_i^{j+1} = \frac{1+2a\tau}{1+2a\tau - \gamma_1\gamma_2} \times$$
$$\times \left[P_i^j + \gamma_1 \cdot Q_{i-1}^{j+1} - \frac{\gamma_1}{1+2a\tau} \cdot Q_i^j - \frac{\gamma_1\gamma_2}{1+2a\tau} \cdot P_{i-1}^{j+1}\right] \tag{16}$$

Combining the explicit (6), (7) and purely implicit scheme (11), (12), we consider a family of schemes defined on a four-point template

$$P_t^n + \frac{c^2}{\overline{F}}\left(\sigma \cdot Q_{\bar{x}}^{n+1} + \left(1-\sigma\right)Q_{\bar{x}}^n\right) = 0, \tag{17}$$

$$Q_t^n + \overline{F}\left(\sigma \cdot P_{\bar{x}}^{n+1} + \left(1-\sigma\right)P_{\bar{x}}^n + 2aQ_i^{n+1}\right) = 0, \tag{18}$$
$$x \in \omega_h, \quad t \in \omega_\tau$$

with initial conditions

$$P\left(0, x_i\right) = P_0\left(x_i\right), \quad Q\left(0, x_i\right) = Q_0\left(x_i\right), \tag{19}$$
$$x_i \in \omega_h,$$

and boundary conditions

$$P\left(t_j, 0\right) = P_0\left(t_j\right), \quad Q\left(t_j, 0\right) = Q_0\left(t_j\right), \quad t_j \in \omega_\tau, \tag{20}$$

$$P\left(t_j, x_{\frac{N_x}{2}}\right) = P\left(t_j, x_{\frac{N_x}{2}-1}\right) + P_{res}, \tag{21}$$

$$Q\left(t_j, x_{\frac{N_x}{2}}\right) = Q\left(t_j, x_{\frac{N_x}{2}-1}\right) + Q_{res}, \tag{22}$$

Scheme (6), (7) and (11), (12) belong to this family and correspond to σ=0 and σ=1, respectively.

Let us calculate the residual for this system of difference equations

$$\psi_1 = P_t^n + \frac{c^2}{\overline{F}}\left(\sigma \cdot Q_{\bar{x}}^{n+1} + \left(1-\sigma\right)Q_{\bar{x}}^n\right), \tag{23}$$

$$\psi_2 = Q_t^n + \overline{F}\left(\sigma \cdot P_{\bar{x}}^{n+1} + \left(1-\sigma\right)P_{\bar{x}}^n + 2aQ_{\bar{x}}^{n+1}\right) \tag{24}$$

We use the Taylor series expansion

$$P\left(t_{n+1}, x_i\right) = P\left(t_{n+\frac{1}{2}}, x_i\right) + \frac{\tau}{2} \cdot \left(\frac{\partial P}{\partial t}\right)\left(t_{n+\frac{1}{2}}, x_i\right) +$$
$$+ \frac{\tau^2}{8} \cdot \left(\frac{\partial^2 P}{\partial t^2}\right)\left(t_{n+\frac{1}{2}}, x_i\right) + \frac{\tau^3}{48} \cdot \left(\frac{\partial^3 P}{\partial t^3}\right)\left(t_{n+\frac{1}{2}}, x_i\right) +$$
$$+ \frac{\tau^4}{384} \cdot \left(\frac{\partial^4 P}{\partial t^4}\right)\left(t_{n+\frac{1}{2}}, x_i\right) + O\left(\tau^5\right),$$

$$P\left(t_n, x_i\right) = P\left(t_{n+\frac{1}{2}}, x_i\right) - \frac{\tau}{2} \cdot \left(\frac{\partial P}{\partial t}\right)\left(t_{n+\frac{1}{2}}, x_i\right) +$$
$$+ \frac{\tau^2}{8} \cdot \left(\frac{\partial^2 P}{\partial t^2}\right)\left(t_{n+\frac{1}{2}}, x_i\right) - \frac{\tau^3}{48} \cdot \left(\frac{\partial^3 P}{\partial t^3}\right)\left(t_{n+\frac{1}{2}}, x_i\right) +$$
$$+ \frac{\tau^4}{384} \cdot \left(\frac{\partial^4 P}{\partial t^4}\right)\left(t_{n+\frac{1}{2}}, x_i\right) + O\left(\tau^5\right)$$

Substituting these expansions into $\frac{(P_i^{n+1} - P_i^n)}{\tau}$ we obtain that

$$P_t^n = \left(\frac{\partial P}{\partial t}\right)\left(t_{n+\frac{1}{2}}, x_i\right) + O\left(\tau^2\right) \tag{25}$$

Similarly

$$Q_t^n = \left(\frac{\partial Q}{\partial t}\right)\left(t_{n+\frac{1}{2}}, x_i\right) + O\left(\tau^2\right) \tag{26}$$

Let us expand the functions P and Q with respect to the variable x:

$$Q_{i-1}^n = Q_i^n - h\left(\frac{\partial Q}{\partial x}\right)_i^n + \frac{h^2}{2}\left(\frac{\partial^2 Q}{\partial x^2}\right)_i^n + O\left(h^3\right)$$

Now let's find the backward difference derivative:

$$\frac{Q_i^n - Q_{i-1}^n}{h} = \left(\frac{\partial Q}{\partial x}\right)_i^n - \frac{h^2}{2}\left(\frac{\partial^2 Q}{\partial x^2}\right)_i^n + O\left(h^2\right) \tag{27}$$

Similarly, we have

$$\frac{P_i^n - P_{i-1}^n}{h} = \left(\frac{\partial P}{\partial x}\right)_i^n - \frac{h^2}{2}\left(\frac{\partial^2 P}{\partial x^2}\right)_i^n + O\left(h^2\right) \tag{28}$$

Substituting (25)-(28) into (23) and (24), we get

$$\psi_1 = \left(\frac{\partial P}{\partial t}\right)_i^{n+\frac{1}{2}} +$$
$$+ \frac{c^2}{\overline{F}}\left\{\sigma \cdot \left[\left(\frac{\partial Q}{\partial x}\right)_i^{n+1} - \frac{h}{2}\left(\frac{\partial^2 Q}{\partial x^2}\right)_i^{n+1} + O\left(h^2\right)\right] + \right.$$
$$\left. + \left(1-\sigma\right)\left[\left(\frac{\partial Q}{\partial x}\right)_i^{n+1} - \frac{h}{2}\left(\frac{\partial^2 Q}{\partial x^2}\right)_i^{n+1} + O\left(h^2\right)\right]\right\} + O\left(\tau^2\right), \tag{29}$$

$$\psi_2 = \left(\frac{\partial Q}{\partial t}\right)_i^{n+\frac{1}{2}} + \overline{F}\left\{\sigma \cdot \left[\left(\frac{\partial P}{\partial x}\right)_i^{n+1} - \frac{h}{2}\left(\frac{\partial^2 P}{\partial x^2}\right)_i^{n+1} + O\left(h^2\right)\right] + \right.$$
$$+ \left(1-\sigma\right)\left[\left(\frac{\partial P}{\partial x}\right)_i^{n+1} - \frac{h}{2}\left(\frac{\partial^2 P}{\partial x^2}\right)_i^{n+1} + O\left(h^2\right)\right] +$$
$$\left. + 2aQ_i^{n+1}\right\} + O\left(\tau^2\right) \tag{30}$$

For the convenience of further calculations for the residuals ψ_1, ψ_2 we set

$$\frac{c^2}{\overline{F}} = 1, \;\; \overline{F} = 1 \tag{31}$$

Let us expand the terms $\left(\frac{\partial Q}{\partial x}\right)_i^{n+1}$, $\left(\frac{\partial Q}{\partial x}\right)_i^{n}$, $\left(\frac{\partial P}{\partial x}\right)_i^{n+1}$, $\left(\frac{\partial P}{\partial x}\right)_i^{n}$ in the Taylor series in t in the neighborhood of the point $t=t_{n+\frac{1}{2}}$

$$\left(\frac{\partial Q}{\partial x}\right)_i^{n+1}=\left(\frac{\partial Q}{\partial x}\right)_i^{n+\frac{1}{2}}+0.5\tau\left(\frac{\partial^2 Q}{\partial x\partial t}\right)_i^{n+\frac{1}{2}}+O\left(\tau^2\right), \qquad (32)$$

$$\left(\frac{\partial Q}{\partial x}\right)_i^{n}=\left(\frac{\partial Q}{\partial x}\right)_i^{n+\frac{1}{2}}-0.5\tau\left(\frac{\partial^2 Q}{\partial x\partial t}\right)_i^{n+\frac{1}{2}}+O\left(\tau^2\right), \qquad (33)$$

$$\left(\frac{\partial P}{\partial x}\right)_i^{n+1}=\left(\frac{\partial P}{\partial x}\right)_i^{n+\frac{1}{2}}+0.5\tau\left(\frac{\partial^2 P}{\partial x\partial t}\right)_i^{n+\frac{1}{2}}+O\left(\tau^2\right) \qquad (34)$$

$$\left(\frac{\partial P}{\partial x}\right)_i^{n}=\left(\frac{\partial P}{\partial x}\right)_i^{n+\frac{1}{2}}-0.5\tau\left(\frac{\partial^2 P}{\partial x\partial t}\right)_i^{n+\frac{1}{2}}+O\left(\tau^2\right) \qquad (35)$$

Then from (29)-(30), taking into account assumption (31), we obtain

$$\begin{aligned}\psi_1=&\left(\frac{\partial P}{\partial t}\right)_i^{n+\frac{1}{2}}+\sigma\cdot\left[\left(\frac{\partial Q}{\partial x}\right)_i^{n+\frac{1}{2}}+0.5\tau\left(\frac{\partial^2 Q}{\partial x\partial t}\right)_i^{n+\frac{1}{2}}+\right.\\&\left.O\left(\tau^2\right)-\frac{h}{2}\left(\frac{\partial^2 Q}{\partial x^2}\right)_i^{n+1}+O\left(h^2\right)\right]+\left(1-\sigma\right)\cdot\left[\left(\frac{\partial Q}{\partial x}\right)_i^{n+\frac{1}{2}}\right.\\&\left.-0.5\tau\left(\frac{\partial^2 Q}{\partial x\partial t}\right)_i^{n+\frac{1}{2}}+O\left(\tau^2\right)-\frac{h}{2}\left(\frac{\partial^2 Q}{\partial x^2}\right)_i^{n}+O\left(h^2\right)\right],\end{aligned} \qquad (36)$$

$$\begin{aligned}\psi_2=&\left(\frac{\partial Q}{\partial t}\right)_i^{n+\frac{1}{2}}+\sigma\cdot\left[\left(\frac{\partial P}{\partial x}\right)_i^{n+\frac{1}{2}}+0.5\tau\left(\frac{\partial^2 P}{\partial x\partial t}\right)_i^{n+\frac{1}{2}}+\right.\\&\left.+O\left(\tau^2\right)-\frac{h}{2}\left(\frac{\partial^2 P}{\partial x^2}\right)_i^{n+1}+O\left(h^2\right)\right]+\left(1-\sigma\right)\cdot\left[\left(\frac{\partial P}{\partial x}\right)_i^{n+\frac{1}{2}}-\right.\\&\left.-0.5\tau\left(\frac{\partial^2 P}{\partial x\partial t}\right)_i^{n+\frac{1}{2}}+O\left(\tau^2\right)-\frac{h}{2}\left(\frac{\partial^2 P}{\partial x^2}\right)_i^{n}+O\left(h^2\right)+2aQ_i^{n+1}\right]\end{aligned} \qquad (37)$$

From the basic equations (1), (2) we have

$$\left(\frac{\partial Q}{\partial x}\right)_i^{n+\frac{1}{2}}=-\left(\frac{\partial P}{\partial t}\right)_i^{n+\frac{1}{2}},$$

$$\left(\frac{\partial P}{\partial x}\right)_i^{n+\frac{1}{2}}=-\left(\frac{\partial Q}{\partial t}\right)_i^{n+\frac{1}{2}}-2aQ_i^{n+\frac{1}{2}}$$

We substitute these expressions into (37) and obtain

$$\begin{aligned}\psi_1=&\left(\frac{\partial P}{\partial t}\right)_i^{n+\frac{1}{2}}-\sigma\cdot\left(\frac{\partial P}{\partial t}\right)_i^{n+\frac{1}{2}}-\left(1-\sigma\right)\cdot\left(\frac{\partial P}{\partial t}\right)_i^{n+\frac{1}{2}}+\\&+0.5\left(2\sigma-1\right)\left(\frac{\partial^2 Q}{\partial x\partial t}\right)_i^{n+\frac{1}{2}}-\frac{\sigma h}{2}\cdot\left(\frac{\partial^2 Q}{\partial x^2}\right)_i^{n+1}-\\&-\frac{\left(1-\sigma\right)h}{2}\cdot\left(\frac{\partial^2 Q}{\partial x^2}\right)_i^{n}+O\left(\tau^2+h^2\right),\end{aligned}$$

$$\begin{aligned}\psi_2=&\left(\frac{\partial Q}{\partial t}\right)_i^{n+\frac{1}{2}}-\sigma\cdot\left(\frac{\partial Q}{\partial t}\right)_i^{n+\frac{1}{2}}-\left(1-\sigma\right)\cdot\left[-\left(\frac{\partial Q}{\partial t}\right)_i^{n+\frac{1}{2}}-2aQ_i^{n+\frac{1}{2}}\right]+\\&+0.5\tau\cdot\left(2\sigma-1\right)\left(\frac{\partial^2 P}{\partial x\partial t}\right)_i^{n+\frac{1}{2}}-\frac{\left(1-\sigma\right)h}{2}\cdot\left(\frac{\partial^2 P}{\partial x^2}\right)_i^{n+1}-\\&-\frac{\left(1-\sigma\right)h}{2}\cdot\left(\frac{\partial^2 P}{\partial x^2}\right)_i^{n}+\left(1-\sigma\right)2aQ_i^{n+\frac{1}{2}}+O\left(\tau^2+h^2\right)\end{aligned}$$

From this it is clear that the terms containing $\left(\frac{\partial P}{\partial t}\right)_i^{n+\frac{1}{2}}$, $\left(\frac{\partial Q}{\partial t}\right)_i^{n+\frac{1}{2}}$ are cancelled out and it turns out that

$$\begin{aligned}\psi_1=&0.5\tau\left(2\sigma-1\right)\left(\frac{\partial^2 Q}{\partial x\partial t}\right)_i^{n+\frac{1}{2}}-\frac{\sigma h}{2}\cdot\left(\frac{\partial^2 Q}{\partial x^2}\right)_i^{n+1}-\\&-\frac{\left(1-\sigma\right)h}{2}\cdot\left(\frac{\partial^2 Q}{\partial x^2}\right)_i^{n}+O\left(\tau^2+h^2\right),\end{aligned} \qquad (38)$$

$$\begin{aligned}\psi_2=&0.5\tau\left(2\sigma-1\right)\left(\frac{\partial^2 P}{\partial x\partial t}\right)_i^{n+\frac{1}{2}}-\frac{\sigma h}{2}\cdot\left(\frac{\partial^2 P}{\partial x^2}\right)_i^{n+1}-\frac{\left(1-\sigma\right)h}{2}\times\\&\times\left(\frac{\partial^2 P}{\partial x^2}\right)_i^{n}+2a\left(1-\sigma\right)\cdot\left(Q_i^{n+1}-Q_i^{n+\frac{1}{2}}\right)+O\left(\tau^2+h^2\right).\end{aligned} \qquad (39)$$

Next, we expand the second derivatives in a Taylor series in t in the neighborhood of the point $t=t_{n+\frac{1}{2}}$

$$\begin{aligned}\left(\frac{\partial^2 Q}{\partial x^2}\right)_i^{n+1}&=\left(\frac{\partial^2 Q}{\partial x^2}\right)_i^{n+\frac{1}{2}}+O\left(\tau\right),\\\left(\frac{\partial^2 Q}{\partial x^2}\right)_i^{n}&=\left(\frac{\partial^2 Q}{\partial x^2}\right)_i^{n+\frac{1}{2}}+O\left(\tau\right),\\\left(\frac{\partial^2 P}{\partial x^2}\right)_i^{n+1}&=\left(\frac{\partial^2 P}{\partial x^2}\right)_i^{n+\frac{1}{2}}+O\left(\tau\right),\\\left(\frac{\partial^2 P}{\partial x^2}\right)_i^{n}&=\left(\frac{\partial^2 P}{\partial x^2}\right)_i^{n+\frac{1}{2}}+O\left(\tau\right)\end{aligned} \qquad (40)$$

Let us assume that $Q_i^{n+1}=Q_i^{n+\frac{1}{2}}+O(\tau^2)$.

Taking into account (40), from (38) and (39) we obtain

$$\psi_1=0.5\tau\left(2\sigma-1\right)\left(\frac{\partial^2 Q}{\partial x\partial t}\right)_i^{n+\frac{1}{2}}-0.5h\cdot\left(\frac{\partial^2 Q}{\partial x^2}\right)_i^{n+\frac{1}{2}}+O\left(\tau^2+h^2\right),$$

$$\psi_2=0.5\tau\left(2\sigma-1\right)\left(\frac{\partial^2 P}{\partial x\partial t}\right)_i^{n+\frac{1}{2}}-0.5h\cdot\left(\frac{\partial^2 P}{\partial x^2}\right)_i^{n+\frac{1}{2}}+O\left(\tau^2+h^2\right)$$

From the system of equations (1), (2) we have that

$$\frac{\partial^2 Q}{\partial x^2}=-\frac{\partial^2 P}{\partial t\partial x},\quad \frac{\partial^2 Q}{\partial x\partial t}=-\frac{\partial^2 P}{\partial x^2}-2a\frac{\partial Q}{\partial x}$$

Let's consider

$$\begin{aligned}\psi_1+\psi_2=&0.5\tau\left(2\sigma-1\right)\left(\frac{\partial^2 Q}{\partial x\partial t}\right)_i^{n+\frac{1}{2}}-0.5h\cdot\left(\frac{\partial^2 P}{\partial x^2}\right)_i^{n+\frac{1}{2}}+\\&+0.5\tau\left(2\sigma-1\right)\left(\frac{\partial^2 P}{\partial x\partial t}\right)_i^{n+\frac{1}{2}}-0.5h\cdot\left(\frac{\partial^2 Q}{\partial x^2}\right)_i^{n+\frac{1}{2}}+O\left(\tau^2+h^2\right)\end{aligned}$$

Because

$$\left(\frac{\partial^2 P}{\partial x^2}\right)_i^{n+\frac{1}{2}} = -\left(\frac{\partial^2 Q}{\partial x \partial t}\right)_i^{n+\frac{1}{2}} - 2a\left(\frac{\partial Q}{\partial x}\right)_i^{n+\frac{1}{2}},$$

$$\left(\frac{\partial^2 Q}{\partial x^2}\right)_i^{n+\frac{1}{2}} = -\left(\frac{\partial^2 P}{\partial x \partial t}\right)_i^{n+\frac{1}{2}}$$

we get that

$$\psi = \psi_1 + \psi_2 = 0.5\tau\left(2\sigma\tau - \tau + h\right)\cdot\left(\frac{\partial^2 Q}{\partial x \partial t}\right)_i^{n+\frac{1}{2}} - ah\cdot\left(\frac{\partial Q}{\partial x}\right)_i^{n+\frac{1}{2}} +$$

$$+0.5\tau\left(2\sigma\tau - \tau + h\right)\cdot\left(\frac{\partial^2 P}{\partial x \partial t}\right)_i^{n+\frac{1}{2}} + O\left(\tau^2 + h^2\right)$$

From this it is clear that the scheme with weights has the second order of approximation

$$\psi = O\left(\tau^2 + h^2\right)$$

If

$$\sigma = \frac{1}{2} - \frac{h}{2\tau} = \sigma_0 \tag{41}$$

and for $\sigma \neq \sigma_0$ – first order, $\psi = O(\tau + h)$.

Thus, the difference scheme with the weight parameter $\sigma = \frac{1}{2} - \frac{h}{2\tau}$ has the second order of approximation $O(\tau^2 + h^2)$ and is stable with respect to the initial condition in the energy norm. In accordance with the Lax theorem, this guarantees the convergence of the scheme and the validity of the obtained numerical results.

2.1. Stability with respect to initial data

Let us now show that the scheme of the family of schemes with weights (23)-(29) is stable with respect to the initial data at

$$\sigma \geq \frac{1}{2} - \frac{h}{2\tau}$$

For proof we use the method of energy inequalities.

$$u_t^n + \sigma v_{\bar{x}}^{n+1} + \left(1 - \sigma\right) v_{\bar{x}}^n = 0, \tag{42}$$

We can consider the sum $w = u + v$

$$w_t^n + \sigma w_{\bar{x}}^{n+1} + \left(1 - \sigma\right) w_{\bar{x}}^n = 0 \tag{43}$$

On the segment $0 \leq x \leq 1$ we introduce the grid $\bar{\omega}_h = \{x_i = ih, i = 0.1, \ldots, N, N_x = 1\}$

The scalar product and the norm are defined as follows:

$$\left(y, z\right] = \sum_{i=1}^{N} \circledcirc y_i \cdot z_i \cdot h, \qquad y = \sqrt{\left(y, z\right]}$$

Considering that

$$y^{n+1} = 0.5\left(y^{n+1} + y^n\right) + 0.5\left(y^{n+1} - y^n\right),$$

$$y^n = 0.5\left(y^{n+1} + y^n\right) - 0.5\left(y^{n+1} - y^n\right),$$

and assuming $y = w_{\bar{x}}$..

Let us rewrite the scheme (43) in the following form

$$w_t + \sigma\left(0.5\left(w_{\bar{x}}^{n+1} + w_{\bar{x}}^n\right) + 0.5\tau \cdot w_{\bar{x}t}^n\right) + 0.5 \times$$

$$\times\left(1 - \sigma\right)\left[\left(w_{\bar{x}}^{n+1} + w_{\bar{x}}^n\right) - \tau w_{\bar{x}t}^n\right] = 0,$$

$$w_t^n + \left(\sigma - 0.5\right)\tau \cdot w_{\bar{x}t}^n + 0.5\left(w_{\bar{x}}^{n+1} + w_{\bar{x}}^n\right) = 0 \tag{44}$$

Let's multiply this equation by $2\tau w_{\bar{x}t}^n = 2\left(w_{\bar{x}}^{n+1} - w_{\bar{x}}^n\right)$

$$2\tau w_t^n \cdot w_{\bar{x}t}^n + 2\tau^2\left(\sigma - 0.5\right)\left(w_{\bar{x}t}^n\right)^2 + \left(w_{\bar{x}}^{n+1}\right)^2 - \left(w_{\bar{x}}^n\right)^2 = 0$$

we transform the first term as follows

$$2w_t^n \cdot w_{\bar{x}t}^n = \left(w_{\bar{x}t}^2\right) + h\left(w_{\bar{x}t}^n\right)^2$$

Then

$$\tau \cdot \left(w_{\bar{x}t}^2\right) + h\tau\left(w_{\bar{x}t}^n\right)^2 + 2\tau^2\left(\sigma - 0.5\right)\left(w_{\bar{x}t}^n\right)^2 + \left(w_{\bar{x}}^{n+1}\right)^2 - \left(w_{\bar{x}}^n\right)^2 = 0$$

if you combine the second and third terms, you get

$$\tau \cdot \left(w_{\bar{x}t}^2\right) + h\tau\left(w_{\bar{x}t}^n\right)^2 + 2\tau^2\left(\sigma - 0.5\right)\left(w_{\bar{x}t}^n\right)^2 + \left(w_{\bar{x}}^{n+1}\right)^2 - \left(w_{\bar{x}}^n\right)^2 = 0$$

We multiply by and sum over all grid nodes $x_i = ih$, $i = 0, 1, \ldots, N$, and obtain

$$\tau\sum_{i=1}^{N}(w_t^2)_{\bar{x},i}h + 2\tau^2\left((\sigma - 0.5)\tau + 0.5h\right)\cdot\left\|w_{\bar{x}t}^n\right\|^2 + \left\|w_{\bar{x}}^{n+1}\right\|^2 = \left\|w_{\bar{x}}^n\right\|^2$$

we write out the difference derivative with respect to x in the first term, then

$$\tau\sum_{i=1}^{N}\left[(w_t^n)_i^2 - (w_t^n)_{i-1}^2\right] + 2\tau\left((\sigma - 0.5)\tau + 0.5h\right)\times$$

$$\times\left\|w_{\bar{x}t}^n\right\|^2 + \left\|w_{\bar{x}}^{n+1}\right\|^2 = \left\|w_{\bar{x}}^n\right\|^2$$

We expand the sum, reduce the terms and get

$$\tau\sum_{i=1}^{N}\left[(w_t^n)_N^2 - (w_t^n)_0^2\right] + 2\tau\left((\sigma - 0.5)\tau + 0.5h\right)\times$$

$$\times\left\|w_{\bar{x}t}^n\right\|^2 + \left\|w_{\bar{x}}^{n+1}\right\|^2 = \left\|w_{\bar{x}}^n\right\|^2$$

here $w_{t,0}^n = w_t(t, 0) = 0$, since $w_t(t, 0) \equiv 0$.

Finally we obtain the identity

$$\tau \cdot (w_t^n)_N^2 + 2\tau\left((\sigma - 0.5)\tau + 0.5h\right)\cdot\left\|w_{\bar{x}t}^n\right\|^2 + \left\|w_{\bar{x}}^{n+1}\right\|^2 = \left\|w_{\bar{x}}^n\right\|^2 \tag{45}$$

From identity (45) it is clear that if

$$\left(\sigma - 0.5\right)\tau + 0.5h \geq 0$$

that is $\sigma = \frac{1}{2} - \frac{h}{2\tau} = \sigma_0$, then

$$\left\|w_{\bar{x}}^{j+1}\right\| \leq \left\|w_{\bar{x}}^j\right\| \leq \ldots \leq \left\|w_{\bar{x}}^0\right\| \tag{46}$$

This inequality proves that scheme (43) is stable with respect to the initial data in the energy norm

$$\left\|w\right\|_{(1)} = \left\|w_{\bar{x}}\right\|$$

3. Statement of the inverse problem

To formulate the inverse problem to (1)-(5), the following additional conditions are set

$$P\left(T, x\right) = P^1\left(x\right), Q\left(T, x\right) = Q^1\left(x\right) \quad \text{at} \quad t = T \tag{47}$$

for pressure and volumetric flow rate of gas.

In the inverse problem, it is necessary to find $P^0(x)$ and $Q^0(x)$ from equation (1)-(2), condition (4)-(5) and additional conditions (47).

4. Statement of the variational problem

One of the fairly common methods for solving inverse problems of mathematical physics is to reduce the problem (1), (2), (4), (5), (47) to an optimal control problem.

It is necessary to minimize the objective functional:

$$J\left(P^0,Q^0\right)=\int_0^{2l}\left[P\left(T,x;P^0(x)\right)-P^1(x)\right]^2dx+$$
$$+\int_0^{2l}\left[Q\left(T,x;Q^0\right)-Q^1\left(x\right)\right]^2dx\to\min \tag{48}$$

We minimize the functional (48) using the gradient iterative method

$$P_{n+1}^0=P_n^0-\alpha\cdot J'\left(P_n^0\right),\;\; Q_{n+1}^0=Q_n^0-\alpha\cdot J'\left(Q_n^0\right) \tag{49}$$

where *α* is the iteration parameter, *n* is the iteration number.

The first variation of the objective functional (48)

$$\delta J\left(P^0,Q^0\right)=J\left(P^0+\delta P^0,Q^0+\delta Q^0\right)-J\left(P^0,Q^0\right)=$$
$$=\int_0^{2l}\left[P\left(T,x;P^0+\delta P^0\right)-P^1\left(x\right)\right]^2dx+$$
$$+\int_0^{2l}\left[Q\left(T,x;Q^0+\delta Q^0\right)-Q^1\left(x\right)\right]^2dx-$$
$$-\int_0^{2l}\left[P\left(T,x;P^0\right)-P^1\left(x\right)\right]^2dx-\int_0^{2l}\left[Q\left(T,x;Q^0\right)-Q^1\left(x\right)\right]^2dx,$$

Because

$$P\left(\mathrm{T},x;P^0+\delta P^0\right)=P\left(\mathrm{T},x;P^0\right)+\delta P\left(\mathrm{T},x;\delta P^0\right),$$
$$Q\left(\mathrm{T},x;Q^0+\delta Q^0\right)=Q\left(\mathrm{T},x;Q^0\right)+\delta Q\left(\mathrm{T},x;\delta Q^0\right),$$

we have

$$\delta J\left(P^0,Q^0\right)=\int_0^{2l}\delta P\left(T,x;\delta P^0\right)\cdot 2\left[P\left(T,x;P^0\right)-P^1\left(x\right)\right]dx+$$
$$+\int_0^{2l}\delta Q\left(T,x;\delta Q^0\right)\cdot 2\left[Q\left(T,x;Q^0\right)-Q^1\left(x\right)\right]dx$$

On the other hand, by the definition of the Frechet derivative

$$\delta J\left(P^0,Q^0\right)=\left\langle J'P^0,\delta P^0\right\rangle+\left\langle J'Q^0,\delta Q^0\right\rangle \tag{50}$$

Let's introduce the notation

$$\tilde{Q}=Q\left(t,x;Q^0+\delta Q^0\right)\quad \tilde{P}=P\left(t,x;P^0+\delta P^0\right),$$
$$Q=Q\left(t,x;Q^0\right),\; P=P\left(t,x;P^0\right),\;\delta Q=\tilde{Q}-Q,\;\delta P=\tilde{P}-P$$

Let us consider the problem perturbed to (1)-(5)

$$\frac{\partial\tilde{P}}{\partial t}=-\frac{c^2}{\overline{F}}\cdot\frac{\partial\tilde{Q}}{\partial x}, \tag{51}$$

$$\frac{\partial\tilde{Q}}{\partial t}=-\overline{F}\cdot\frac{\partial\tilde{P}}{\partial x}-2a\cdot\tilde{Q}, \tag{52}$$

initial conditions

$$\tilde{P}\left(0,x\right)=P^0\left(x\right)+\delta P^0\left(x\right),\quad \tilde{Q}\left(0,x\right)=Q^0\left(x\right)+\delta Q^0\left(x\right) \tag{53}$$

and boundary conditions

$$\tilde{P}\left(t,0\right)=P_0\left(t\right),\quad \tilde{Q}\left(t,0\right)=Q_0\left(t\right), \tag{54}$$

$$\tilde{P}\left(t,l+0\right)=\tilde{P}\left(t,l-0\right)+P_{res}\left(t\right),$$
$$\tilde{Q}\left(t,l+0\right)=\tilde{Q}\left(t,l-0\right)+Q_{res}\left(t\right) \tag{55}$$

To obtain the problem for the disturbances $\delta P(T,x;\delta P^0)$ and $\delta Q(T,x;\delta Q^0)$ from problem (51)-(55) we subtract problem (1)-(5) due to the linearity of the equation we have

$$\frac{\partial\delta P}{\partial t}=-\frac{c^2}{\overline{F}}\cdot\frac{\partial\delta Q}{\partial x}, \tag{56}$$

$$\frac{\partial\delta Q}{\partial t}=-\overline{F}\cdot\frac{\partial\delta P}{\partial x}-2a\cdot\delta Q, \tag{57}$$

the initial conditions will take the form

$$\delta P\left(0,x\right)=\delta P^0,\quad \delta Q\left(0,x\right)=\delta Q^0 \tag{58}$$

and boundary conditions

$$\delta P\left(t,0\right)=0,\;\;\delta Q\left(t,0\right)=0 \tag{59}$$

$$\delta P\left(t,l+0\right)=\delta P\left(t,l-0\right),\;\;\delta Q\left(t,l+0\right)=\delta Q\left(t,l-0\right) \tag{60}$$

We multiply (56) by the still unknown function *P**(*t,x*), (57) by *Q**(*t,x*) and integrate over *t* from 0 to *T*, over x from 0 to 2l and sum. As a result, the expression identically equal to zero

$$\left(A\delta P,P^*\right)+\left(B\delta Q,Q^*\right)=\int_0^T\int_0^{2l}\left[\frac{\partial\delta P}{\partial t}+\frac{c^2}{\overline{F}}\cdot\frac{\partial\delta Q}{\partial x}\right]\cdot P^*dxdt+$$
$$+\int_0^T\int_0^{2l}\left[\frac{\partial\delta Q}{\partial t}+\overline{F}\cdot\frac{\partial\delta P}{\partial x}+2a\cdot\delta Q\right]\cdot Q^*dxdt\equiv 0,$$

where $AP=\dfrac{\partial P}{\partial t}+\dfrac{c^2}{\overline{F}}\cdot\dfrac{\partial Q}{\partial x},\;\; BQ=\dfrac{\partial Q}{\partial t}+\overline{F}\cdot\dfrac{\partial P}{\partial x}+2aQ$

We integrate this expression by parts

$$\left(A\delta P,P^*\right)+\left(B\delta Q,Q^*\right)=$$
$$=\int_0^{2l}\left[\delta P\cdot P^*\,|_0^T-\int_0^T\delta P\cdot\frac{\partial P^*}{\partial t}dt\right]dx+\frac{c^2}{\overline{F}}\int_0^T\left[\delta Q\cdot P^*\,|_0^{2l}-\int_0^{2l}\delta Q\cdot\frac{\partial P^*}{\partial x}dx\right]dt+$$
$$+\int_0^{2l}\left[\delta Q\cdot Q^*\,|_0^T-\int_0^T\delta Q\cdot\frac{\partial Q^*}{\partial t}dt\right]dx+\overline{F}\int_0^T\left[\delta P\cdot Q^*\,|_0^{2l}-\int_0^{2l}\delta P\cdot\frac{\partial Q^*}{\partial x}dx\right]dt+$$
$$+2a\int_0^{2l}\int_0^T\delta Q\cdot Q^*dtdx=-\int_0^T\int_0^{2l}\left[\frac{\partial P^*}{\partial t}+\overline{F}\cdot\frac{\partial Q^*}{\partial x}\right]\delta Pdxdt-$$
$$-\int_0^T\int_0^{2l}\left[\frac{\partial Q^*}{\partial t}+\frac{c^2}{\overline{F}}\cdot\frac{\partial P^*}{\partial x}-2a\cdot\delta Q^*\right]\delta Qdxdt+$$
$$+\int_0^{2l}\left[\delta P\left(T,x\right)\cdot P^*\left(T,x\right)-\delta P\left(0,x\right)\cdot P^*\left(0,x\right)\right]dx+$$
$$+\frac{c^2}{\overline{F}}\int_0^T\left[\delta Q\cdot P^*\,|_0^l+\delta Q\cdot P^*\,|_l^{2l}\right]dt+$$
$$+\int_0^{2l}\left[\delta Q\left(T,x\right)\cdot Q^*\left(T,x\right)-\delta Q\left(0,x\right)\cdot Q^*\left(0,x\right)\right]dx+$$
$$+\overline{F}\int_0^T\left[\delta P\cdot Q^*\,|_0^l+\delta P\cdot Q^*\,|_l^{2l}\right]dt.$$

$$\left(A\delta P,P^*\right)+\left(B\delta Q,Q^*\right)=\int_0^T\int_0^{2l}\left[\frac{\partial P^*}{\partial t}+\overline{F}\cdot\frac{\partial Q^*}{\partial x}\right]\delta Pdxdt-$$
$$-\int_0^T\int_0^{2l}\left[\frac{\partial Q^*}{\partial t}+\frac{c^2}{\overline{F}}\cdot\frac{\partial P^*}{\partial x}-2a\cdot Q^*\right]\delta Qdxdt+$$
$$+\int_0^{2l}\left[\delta P\left(T,x\right)\cdot P^*\left(T,x\right)-\delta P^0\cdot P^*\left(0,x\right)\right]dx+$$
$$+\frac{c^2}{\overline{F}}\int_0^T\left[\delta Q\left(t,l-0\right)\cdot P^*\left(t,l-0\right)-\delta Q\left(t,0\right)\cdot P^*\left(t,0\right)+\right.$$
$$\left.+\delta Q\left(t,2l\right)\cdot P^*\left(t,2l\right)-\delta Q\left(t,l+0\right)\cdot P^*\left(t,l+0\right)\right]dx+$$
$$+\int_0^{2l}\left[\delta Q\left(T,x\right)\cdot Q^*\left(T,x\right)-\delta Q^0\cdot Q^*\left(0,x\right)\right]dx+$$
$$+\overline{F}\int_0^T\left[\delta P\left(t,l-0\right)\cdot Q^*\left(t,l-0\right)-\delta P\left(t,0\right)\cdot Q^*\left(t,0\right)+\right.$$
$$\left.+\delta P\left(t,2l\right)\cdot Q^*\left(t,2l\right)-\delta P\left(t,l+0\right)\cdot Q^*\left(t,l+0\right)\right]dt. \tag{61}$$

In the last expression (61), the terms outside the double integral with the factors $\delta P(t,0)$ and $\delta Q(t,0)$ are equal to zero, according to conditions (59). Due to the fulfillment of the matching condition (60) and the required additional condition at $x=l$ (at the coalface)

$$P^*\left(t,l-0\right)=P^*\left(t,l+0\right),\quad Q^*\left(t,l-0\right)=Q^*\left(t,l+0\right) \qquad (62)$$

it turns out that

$$\delta Q\left(t,l-0\right)\cdot P^*\left(t,l-0\right)=\delta Q\left(t,l+0\right)\cdot P^*\left(t,l+0\right), \qquad (63)$$

$$\delta P\left(t,l-0\right)\cdot Q^*\left(t,l-0\right)=\delta P\left(t,l+0\right)\cdot Q^*\left(t,l+0\right) \qquad (64)$$

From physical considerations, we assume that at the wellhead, the perturbations of pressure $\delta P(t,2l)$ and volumetric gas flow rate $\delta Q(t,2l)$ are negligibly small. Now the terms contained under the integral over x remain.

$$\int_0^{2l}\delta P\left(T,x\right)\cdot P^*\left(T,x\right)dx+\int_0^{2l}\delta Q\left(T,x\right)\cdot Q^*\left(T,x\right)dx-$$
$$-\int_0^{2l}\delta P^0\cdot P^*\left(0,x\right)dx+\int_0^{2l}\delta Q^0\cdot Q^*\left(0,x\right)dx=0$$

Let us introduce the notation of the operators

$$A^*P^*=\frac{\partial P^*}{\partial t}+\overline{F}\cdot\frac{\partial Q^*}{\partial x},\quad B^*Q^*=\frac{\partial Q^*}{\partial t}+\frac{c^2}{\overline{F}}\cdot\frac{\partial P^*}{\partial x}-2a\cdot Q^* \qquad (65)$$

The fulfillment of this equality leads to the following lemma.

Lemma 1 Let P^0, $P+\delta P^0\in Q_{ad}$, Q^0, $Q+\delta Q^0\in Q_{ad}$ be elements belonging to the domain of possible solutions. If $P(t,x;P^0(x))$, $Q(t,x;Q^0(x))$ is a solution to problem (1)-(5) and the Lagrange integral identity is satisfied

$$\left(AP,P^*\right)+\left(BQ,Q^*\right)=\left(P,A^*P^*\right)+\left(Q,B^*Q^*\right)$$

then it takes place

$$\int_0^{2l}\delta P\left(T,x\right)\cdot P^*\left(T,x\right)dx+\int_0^{2l}\delta Q\left(T,x\right)\cdot Q^*\left(T,x\right)dx=$$
$$=\int_0^{2l}\delta P^0\cdot P^*\left(0,x\right)dx+\int_0^{2l}\delta Q^0\cdot Q^*\left(0,x\right)dx \qquad (66)$$

Condition (66), taking into account the boundary conditions (58), the first variation of the functional $\delta J(P^0,Q^0)$ and the definition of the Frechet derivative (50), will take the form

$$\left\langle\delta P\left(T,x\right),2\left[P\left(T,x;P^0\right)-P^1\left(x\right)\right]\right\rangle+$$
$$+\left\langle\delta Q\left(T,x\right),2\left[Q\left(T,x;Q^0\right)-Q^1\left(x\right)\right]\right\rangle=$$
$$=\left\langle J'P^0,\delta P^0\right\rangle+\left\langle J'Q^0,\delta Q^0\right\rangle \qquad (67)$$

These conditions follow from the requirements of fulfilling the Lagrange identity.

5. Statement of the conjugate problem

The fulfillment of the Lagrange identity, the requirement that all non-integral terms be equal to zero, and the conditions of Lemma 1 lead to the following conjugate problem

$$\frac{\partial P^*}{\partial t}+\overline{F}\cdot\frac{\partial Q^*}{\partial x}=0, \qquad (68)$$

$$\frac{\partial Q^*}{\partial t}+\frac{c^2}{\overline{F}}\cdot\frac{\partial P^*}{\partial x}-2a\cdot Q^*=0, \qquad (69)$$

$$P^*\left(t,0\right)=0,Q^*\left(t,0\right)=0, \qquad (70)$$

$$P^*\left(T,x\right)=2\left[P\left(T,x;P^0\right)-P^1\right],$$
$$Q^*\left(T,x\right)=2\left[Q\left(T,x;Q^0\right)-Q^1\right] \qquad (71)$$

The above calculations of the solution of the variational problem prove the following theorem.

Theorem 1 (on the convergence of the gradient method in the gas lift problem).

Let the functional

$$J\left[P^0,Q^0\right]=\int_0^{2l}\left[P\left(T,x;P^0\right)-P^1\left(x\right)\right]^2dx+$$
$$+\int_0^{2l}\left[Q\left(T,x;Q^0\right)-Q^1\left(x\right)\right]^2dx$$

be defined on a convex and bounded set of admissible values

$$P^0,Q^0\in G_{ad}\subset W_2^1\left(0,2l\right)$$

be continuous and convex on G_{ad}. Then the sequence $\{P_n^0,\ Q_n^0\}$, constructed using the iterative gradient method, satisfies the relations:

$$\lim_{n\to\infty}J\left[P_n^0,Q_n^0\right]=J_*,\quad \lim_{n\to\infty}\|J'\left[P_n^0,Q_n^0\right]\|=0$$

and there is a constant $C>0$ such that:

$$0\le\left|J\left[P_n^0,Q_n^0\right]-J_*\right|\le\frac{C}{n}$$

Here J_* is the minimal value of the functional, and the gradients $J'[P^0]$, $J'[Q^0]$ are calculated using the formulas:

$$\begin{cases}J'\left(P_n\right)=-P^*\left(0,x\right),\\ J'\left(Q_n\right)=-Q^*\left(0,x\right)\end{cases}$$

where $P^*(0,x)$ and $Q^*(0,x)$ are the solution to the adjoint problem.

Thus, all the conditions for the convergence of the gradient method on a convex bounded set in a Hilbert space are satisfied.

6. Algorithm for solving a variational problem

1. We set the initial approximation P^0, Q^0.

2. Let's assume that P_n^0, Q_n^0 is already known, then we solve the direct problem:

$$\frac{\partial P}{\partial t}=-\frac{c^2}{\overline{F}}\cdot\frac{\partial Q}{\partial x},\quad t\ge0,x\in\left(0,2l\right),$$
$$\frac{\partial Q}{\partial t}=-\overline{F}\cdot\frac{\partial P}{\partial x}-2a\cdot Q,\quad t\ge0,x\in\left(0,2l\right),$$
$$P\left(0,x\right)=P^0\left(x\right),\quad Q\left(0,x\right)=Q^0\left(x\right),$$
$$P\left(t,0\right)=P_0\left(t\right),\quad P\left(t,0\right)=P_0\left(t\right),$$
$$P\left(t,l+0\right)=P_0\left(t,l-0\right)+P_{res}\left(t\right),$$
$$Q\left(t,l+0\right)=Q_0\left(t,l-0\right)+Q_{res}\left(t\right) \qquad (72)$$

3. Calculate the approximate value of the functional using the quadrature formula

$$J\left(P_n^0,Q_n^0\right)=\int_0^{2l}\left[P\left(T,x;P_n^0\right)-P^1\left(x\right)\right]^2dx+$$
$$+\int_0^{2l}\left[P\left(T,x;Q_n^0\right)-Q^1\left(x\right)\right]^2dx \qquad (73)$$

4. If the current value of the norm of the functional $J(P_n^0,\ Q_n^0)$ is not small enough, then we solve the conjugate problem:

$$\frac{\partial P^*}{\partial t}+\overline{F}\cdot\frac{\partial Q^*}{\partial x}=0,\quad \frac{\partial Q^*}{\partial t}+\frac{c^2}{\overline{F}}\cdot\frac{\partial P^*}{\partial x}+2a\cdot Q^*=0$$
$$P^*(t,0)=0,Q^*(t,0)=0,\quad P^*(T,x)=2\left[P(\mathrm{T},x;P^0)-P^1\right]$$
$$Q^*(T,x)=2\left[Q(\mathrm{T},x;Q^0)-Q^1\right] \tag{74}$$

5. From the solution P^*, Q^* of the conjugate problem (74) for P and Q we determine the gradient of the functional

$$\begin{cases} J'(P_n)=-P^*(0,x), \\ J'(Q_n)=-Q^*(0,x) \end{cases} \tag{75}$$

6. The following approximations of the initial conditions for $P(0,x)$ and $Q(0,x)$ are found using the formulas

$$P^0_{n+1}(x)=P^0_n(x)-\alpha\cdot J'(P_n),$$
$$Q^0_{n+1}(x)=Q^0_n(x)-\alpha\cdot J'(Q_n) \tag{76}$$

7. Let's move on to point 3.

7. Numerical implementation of the algorithm for solving a variational problem

7.1. Difference scheme for solving a direct problem

Let us approximate the direct problem (1)-(5). Let N_t be the number of nodes of the uniform grid on the interval $[0,T]$, and N_x be the number of nodes of the uniform grid on the interval $[0,2l]$.

Let us construct in the domain $\Omega_h=((0,2l)\times(0,T))$ a grid ω_h with a step $h=2l/N_x$, $\tau=T/N_t$, where N_x, N_t are positive integers.

Then in the grid $\omega_h=\{x=ih,\ t=k\tau,\ i=0,1,\ldots,N_x,\ k=0,1,\ldots,N_t\}$ we write the corresponding difference direct problem. Thus, problem (1)-(5) has the following form:

$$\frac{P_i^{k+1}-P_i^k}{\tau}=-\frac{c^2}{\overline{F}}\cdot\frac{Q_i^k-Q_{i-1}^k}{h}, \tag{77}$$

$$\frac{Q_i^{k+1}-Q_i^k}{\tau}=-\overline{F}\cdot\frac{P_i^k-P_{i-1}^k}{h}-2a\cdot Q_i^{k+1}, \tag{78}$$
$$i=0,1,\ldots,N_x,\quad k=0,1,\ldots,N_t,$$

initial conditions

$$P_i^0=P^0(x_i),\quad Q_i^0=Q^0(x_i),\quad i=0,1,\ldots,N_x, \tag{79}$$

and boundary conditions respectively

$$P_0^k=P_0,\quad Q_0^k=Q_0,\quad k=0,1,\ldots,N_t \tag{80}$$

$$P^k_{\frac{N_x}{2}}=P^k_{\frac{N_x}{2}-1}+P_{res},\quad Q^k_{\frac{N_x}{2}}=Q^k_{\frac{N_x}{2}-1}+Q_{res},\quad k=0,1,\ldots,N_t \tag{81}$$

7.2. Difference scheme for solving the conjugate problem

In the same grid domain we write the corresponding difference conjugate problem. Thus, the difference analogue of problem (68)-(71) has the following form:

$$\frac{P_i^{*k+1}-P_i^{*k}}{\tau}+\overline{F}_i\frac{Q_i^{*k+1}-Q_{i-1}^{*k+1}}{h}=0, \tag{82}$$

$$\frac{Q_i^{*k+1}-Q_i^{*k}}{\tau}+\frac{c_i^2}{\overline{F}_i}\cdot\frac{P_i^{*k+1}-P_{i-1}^{*k+1}}{h}+2a\cdot Q_i^{*k+1}=0, \tag{83}$$
$$i=0,1,\ldots,N_x,\qquad k=N_t,\ldots,1,0,$$

$$P_0^{*k}=0,\quad Q_0^{*k}=0,\qquad k=N_t,N_t-1,\ldots,0, \tag{84}$$

$$P_i^{*N_t}=2\cdot\left[P_i^{N_t}-P^{(1)}\right],\ Q_i^{*N_t}=2\cdot\left[Q_i^{N_t}-Q^{(1)}\right], \tag{85}$$
$$i=0,1,\ldots,N_x-1$$

8. Numerical results

Numerical calculations according to the algorithm developed above were carried out with the initial data from the work [9]. The volumetric flow rate of the injected gas was chosen to be Q^0=0.21 m^3/c, and the initial pressure P^0=5177500 Pa, the well depth l=1485 m, the speed of sound in the annular space c_1=331 m/s, the speed of sound in the production well c_2=850 m/s. The cross-sectional area of the annular space of the well $F_1=\pi r_1^2$ m^2, the cross-sectional area of the inner well $F_2=\pi r_2^2$ m^2, r_1=0.06765 m, r_2=0.0365 m. Hydraulic resistance in the annulus λ_1=0.01, hydraulic resistance in the well λ_2=0.23. Gas density ρ_1=0.75 kg/m^3, oil density ρ_2=700 kg/m^3, acceleration due to gravity = 9.8 m/c^2.

The average cross-sectional velocity of the mixture in the ring is $w_1=\dfrac{Q_0(x)}{F_1\rho_1}$ m/s, the average cross-sectional velocity of movement in the well is $w_2=\dfrac{Q_0(x)}{F_2\rho_2}$ m/s.

The peculiarity of the problem being solved is that the coefficients $c(x)$, $\overline{F}(x)$, $a(x)$ have discontinuities at the point $x=l$ and the values are large numbers. Therefore, in the calculations, the initial pressure and the initial volume of the injected gas were specified as linear functions

$$Q_i^0=Q^0+0.5\cdot\left(Q_{\mathrm{out}}-Q^0\right)\cdot x_i/l,$$
$$P_i^0=P^0+0.5\cdot\left(P_{\mathrm{out}}-P^0\right)\cdot x_i/l$$

where Q_{out}, P_{out} are the values of the output volume of the mixture and pressure.

The coefficients $c(x_i)$, $\overline{F}(x_i)$, $a(x_i)$ have a large spread of values, therefore, to ensure a stable calculation, they were normalized as follows

$$a(x_i)=0.5\cdot\left(a(x_i)+a(x_{N_x})-2\cdot a(x_0)\right)/\left(a(x_{N_x})-a(x_0)\right)$$

where

$$a(x_i)=\begin{cases}\dfrac{g}{2w_1}+\dfrac{\lambda_1 w_1}{4d_1}, & i=0,1,\ \ldots,N_x/2,\\[2mm] \dfrac{g}{2w_2}+\dfrac{\lambda_2 w_2}{4d_2}, & i=N_x/2+1,\ \ldots,N_x.\end{cases}$$

The graphs of the functions $a(x)$, $c(x)$, $\overline{F}(x)$ are shown in figure 1.

Using the developed algorithm, numerical calculations were performed in a wide range of input parameters. Difference schemes were used on grids of 50×50, 100×100 sizes. The iteration parameter α=0.009.

Additional conditions for $P^1(x)$, $Q^1(x)$ were specified in the form of parabolic functions

$$Q^1(T,x)=-x^2+b_q\cdot x+c_q,$$
$$P^1(T,x)=-x^2+b_p\cdot x+c_p,$$

where

$$c_q=Q^0,\ c_q=P^0,\ b_q=\frac{Q_{out}-Q^0+4l^2}{2l},\ b_p=\frac{P_{\mathrm{out}}-P^0+4l^2}{2l}$$

Figures 2 and 3 show the graphs of the function $P^0(x)$, $Q^0(x)$. These are the initial data; during the iteration process they will change depending on the additional conditions $P^1(x)$, $Q^1(x)$.

The graphs of $P^1(x)$, $Q^1(x)$ are shown in figures 4 and 5.

In the conducted iterative process, the value of the functional J monotonically decreases and reaches the value $\|J\| \le \varepsilon$, $\varepsilon = 0.001$ at $n=164$ iterations. The graph of the decrease in the value of the functional is shown in figure 6.

Numerical calculations show that the iterative process used to find the pressure value P_n^0 and the volumetric gas flow rate Q_n^0 at $t=0$ converges. The numerical values of the norm of the functional J monotonically decrease and are limited, so the calculated values of P_n^0 and Q_n^0 tend to a parabolic function (fig. 7, 8). This is plausible, since the additional conditions $P^1(x)$, $Q^1(x)$ we specified are parabolic functions. Also, figures 9 and 10 show three-dimensional graphs of $P(t,x)$ and $Q(t,x)$.

Figures 11 and 12 show a comparison of the initial, reconstructed, and accurate pressure and volumetric flow rate. This helps to visually evaluate the effectiveness of the inverse method in the problem of reconstructing the initial conditions for $P^0(x)$ and $Q^0(x)$.

It is known that the conjugate problem carries valuable information about the solution of the direct problem. This property is confirmed by numerical calculations, since the gradients of the functional for determining the initial conditions of the direct problem at each iteration were chosen as the solution of the conjugate problem at $t=0$, i.e.

$$J'\left(P_n^0\right) = P^*\left(0,x\right), J'\left(Q_n^0\right) = Q^*\left(0,x\right)$$

The numerical calculations performed confirm the effectiveness of the proposed algorithm for modeling the gas lift process of oil production.

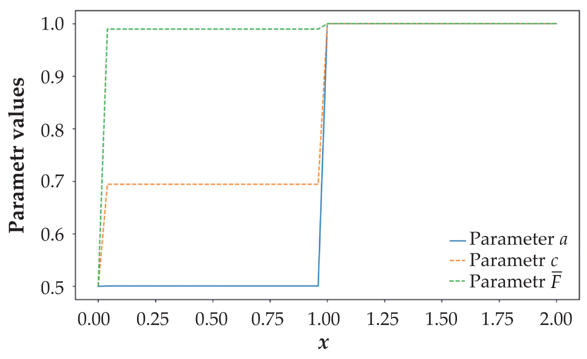

Fig. 1. Graphs of functions a, c, and $\overline{F}$

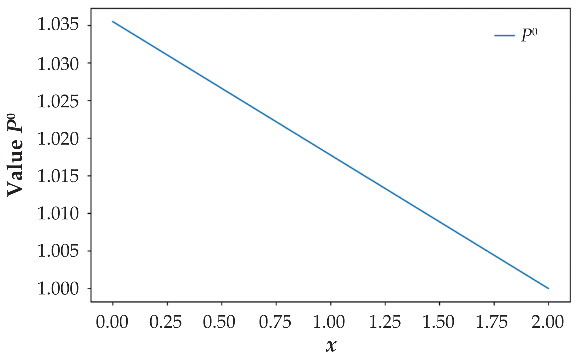

Fig. 2. Graph of functions $P^0(x)$

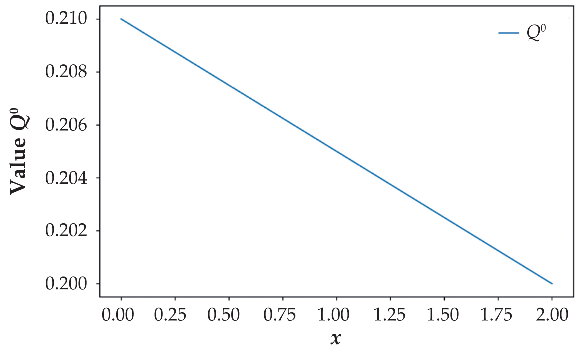

Fig. 3. Graph of functions $Q^0(x)$

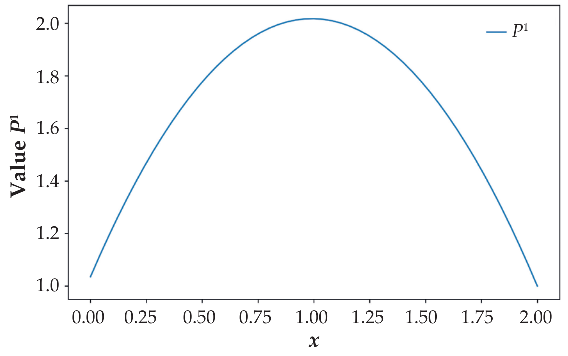

Fig. 4. Graph of functions $P^1(x)$

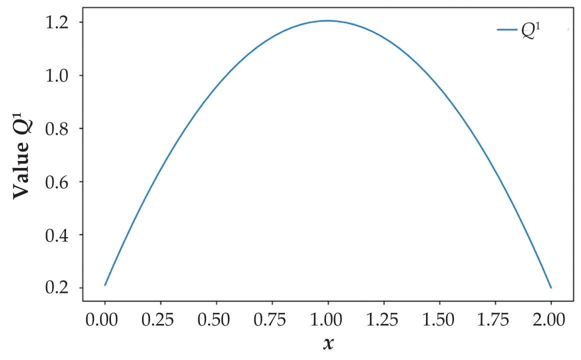

Fig. 5. Graph of functions $Q^1(x)$

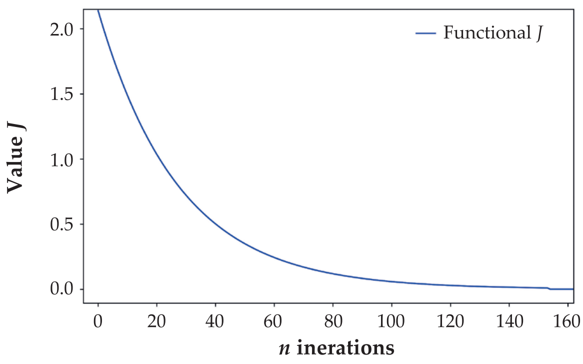

Fig. 6. Graph of decreasing functional J

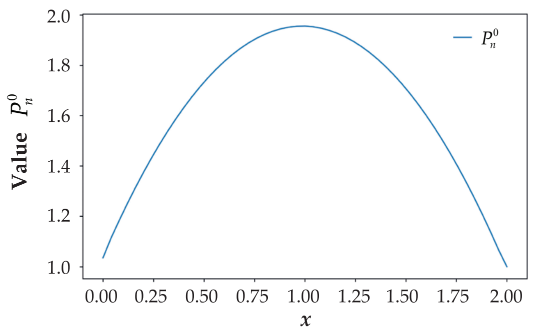

Fig. 7. P_n^0 **graph**

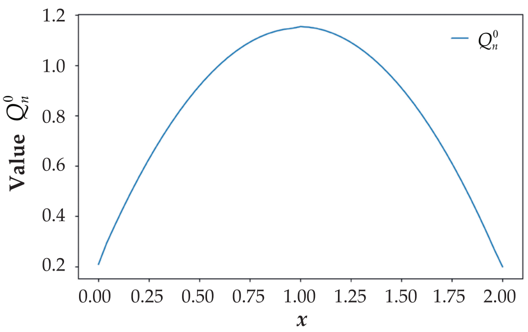

Fig. 8. Q_n^0 **graph**

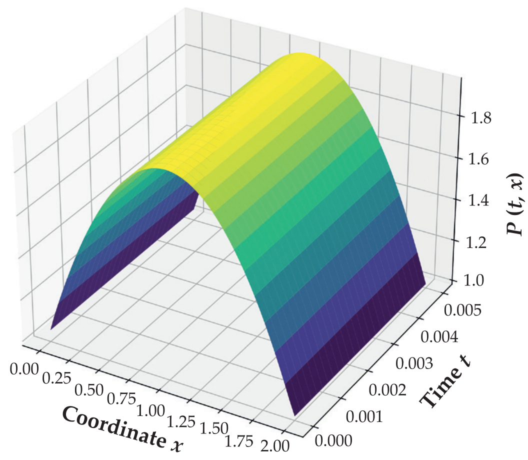

Fig. 9. 3D pressure graph $P(t, x)$

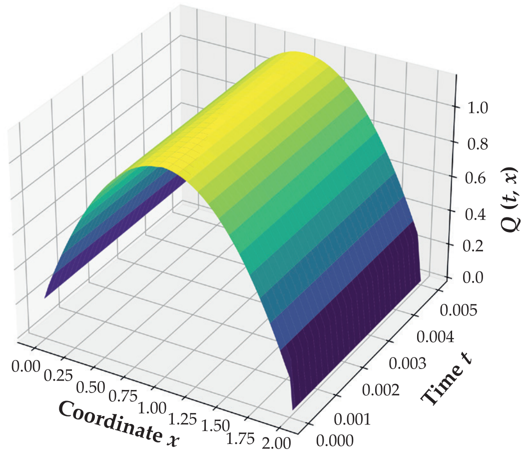

Fig. 10. 3D pressure graph $Q(t, x)$

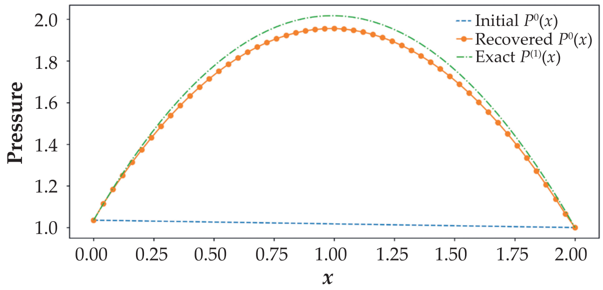

Fig. 11. Comparison of pressure

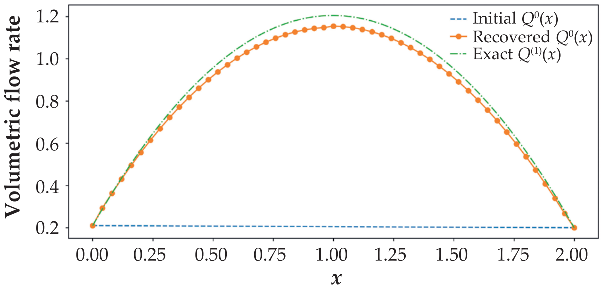

Fig. 12. Comparison of volumetric flow rate

Conclusions

This paper reviews engineering models [56, 57] used to study the gas lift performance curve. These models use Darcy's law and solve a one-dimensional stationary problem. The results are functions calculated by genetic algorithms that describe the gas lift performance curve. In this paper, non-stationary linear Navier-Stokes controls of compressible gas are used to describe the gas-lift process of oil production. The input, bottomhole pressure and volume of the gas-liquid mixture are determined by the method of conjugate equations. The conjugate problem contains valuable information about the solution of the direct problem. The wellhead pressure and volume of the gas-liquid mixture are specified as additional information.

Numerous calculations have been performed in a wide range of input parameters. The compiled program code in the PYTHON language allows for numerical modeling of the gas lift process and finding the optimal operating mode. The gradient method used to determine the pressure and volumetric flow rate of gas at the input converges monotonically.

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