

IMPLICIT NUMERICAL FINITE ELEMENT METHOD FOR A NONEQUILIBRIUM FILTRATION MODEL

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ABSTRACT

Two-phase flow in porous media is an important problem in petroleum engineering, hydrology, and geotechnics. This paper investigates the Muskat–Leverett model for simultaneous water and oil flow, incorporating dynamic capillary pressure to describe nonequilibrium effects during changes in saturation. A weak formulation is presented within an established mixed finite element framework. A fully implicit finite element scheme couples the phase mass balances with the dynamic capillary relation, solving the saturation and pressure equations simultaneously. The computational algorithm is implemented on the FEniCS (dolfin) platform, and the coupled nonlinear system is solved using Newton's method at each time step. Numerical verification is performed using a benchmark problem with a known analytical solution. Computational experiments investigate saturation and pressure profiles, approximation errors under mesh refinement, and the influence of the relaxation parameter τ for values of 1, 0.1, and 0.01. The fully implicit computations reproduce the qualitative features of the analytical solution and demonstrate mesh convergence. Comparison with an IMPES implementation, in which pressure is treated implicitly and saturation explicitly, shows difficulties in accurately reproducing the nonequilibrium solution under the tested conditions. The study provides a computational implementation and comparative assessment of the model, with the comparison conclusions restricted to the algorithms and parameters considered.

Keywords: Muskat-Leverett model for nonequilibrium filtration; nonlinear equation; mixed finite elements; dynamic capillary pressure.

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1. Introduction

In the equilibrium case, the algebraic relationship between the phase pressures and the capillary pressure, which depends on saturation, is expressed as:

$$p_o - p_w = p_c(s)$$

which is well known among standard models in porous media [1-3]. Numerical methods for classical models have been the subject of extensive investigation in recent studies.

In [4], an efficient numerical method was proposed for solving the initial-boundary value problem for a coupled system of equations consisting of a nonlinear parabolic integro-differential partial differential equation and an elliptic equation with a nonlinear term [5, 6].

In [7], a fully discrete numerical scheme is investigated for a model describing two-phase immiscible flow in porous

media with dynamic effects associated with capillary pressure. For time discretization, the implicit Euler method is employed. Spatial discretization is based on the mixed finite element method (MFEM), specifically using lowest-order Raviart-Thomas finite elements. A priori error estimates are obtained for saturation, fluxes, and phase pressures in $L_\infty(0, T; L_2(\Omega))$ for both temporal and spatial discretizations. Using these estimates, the convergence of the scheme is demonstrated. Numerical results are provided to confirm the theoretical findings.

In [8], a numerical modeling scheme is presented for reactive transport of contaminants with nonequilibrium sorption in porous media, using the mixed finite element method combined with the Newton method.

In [9-11], the authors analyze finite volume methods. In [9], a new hybrid algorithm is presented that combines pore-scale descriptions with Darcy's law for two-phase flow in porous media. The liquid volume method is used to model the evolution of the fluid interface. In [10], the main

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computational contribution is a mixed-dimensional finite volume discretization based on a moving mesh. This moving mesh finite volume method (FVMM) tracks fracture movements as a family of connected mesh faces. In particular, the method allows arbitrary movement of triangulation faces while preserving the mass conservation constraint. In [11], a vertex-centered finite volume method is introduced for fully coupled, fully implicit discretization of two-phase flow in fractured porous media. Fractures are modeled discretely as lower-dimensional elements. The method operates on unstructured, locally refined meshes and on parallel computers with distributed memory. Implicit time discretization is employed, and the resulting nonlinear systems of equations are solved using a parallel Newton method.

In [12], a hybrid numerical method is proposed for solving a model problem of two-phase nonequilibrium flow of incompressible fluids in porous media. This problem is relevant in the modern theory of multiphase fluid flow in porous media and has numerous applications. The model under study is based on the assumption that the relative phase permeabilities and capillary pressure depend not only on saturation but also on its time derivative. To solve the problem, the finite volume element method is employed, which is constructed based on integral balance equations, and the approximate solution is chosen from the finite element space.

In the work of S. M. Hassanizadeh and William G. Gray [13], microscopic mass and momentum balance equations for the phases and phase interfaces are used to develop an understanding of capillary pressure at the microscale. The limitations of the standard theory, which defines capillary pressure at the macroscale, are discussed. Finally, an approach is presented demonstrating that capillary pressure is an intrinsic property of the system under study. In particular, it is shown that the presence of phase interfaces and their distribution within a multiphase system are important for describing the system's state.

The thermodynamic approach to defining capillary pressure represents a theoretically justified alternative to defining capillary pressure simply as a hysteresis function of saturation.

In the monograph by B. T. Zhumagulov and V. N. Monakhov [14], the main models of subsurface hydrodynamics are studied both theoretically and numerically - the isothermal and thermal Muskat-Leverett models, as well as the Navier-Stokes model. In the Muskat-Leverett model, a new sought function, the «reduced» pressure, is introduced, which allows obtaining a quasi-linear system consisting of a uniformly elliptic equation for pressure and a parabolic equation for saturation. The book demonstrates that models based on Darcy's law are a simplification of the Navier-Stokes system. Therefore, for a more in-depth study of filtration flows, the Navier-Stokes system can be employed.

In [15], N. T. Danaev, Sh. S. Smagulov and N. M. Temirbekov consider fluid flow through porous media in curvilinear channels with a porous insert. This problem is solved using the Navier-Stokes equations in arbitrary orthogonal curvilinear coordinates. A finite difference scheme and a numerical implementation algorithm are constructed, for which the number of iterations depends weakly on the magnitude of the discontinuities in the equation coefficients. Extensive calculations of fluid flow through porous media are carried out over a wide range of input parameters.

In [16], a new modification of the fictitious domain method is proposed for the numerical simulation of problems governed by the Navier-Stokes equations. For the auxiliary problem, Dirichlet boundary conditions are imposed on the pressure, and the tangential component of velocity is set to zero, which allows formulating a Dirichlet problem for determining the pressure.

In the work of H. Chen and S. Sun [17], a new efficient IMPES scheme is proposed for simulating incompressible, immiscible two-phase flow in heterogeneous porous media with capillary pressure. Compared to traditional IMPES schemes, the proposed scheme preserves physical properties; in particular, the new algorithm locally conserves the mass of both phases. Another attractive feature is that the total velocity remains continuous in the normal direction.

In another work by the same authors [18], based on several types of new semi-implicit IMPES schemes for solving two-phase flow problems in porous media, published by them in recent years, a new framework is proposed for deriving the pressure equation in IMPES. Specifically, the volume conservation equation for each phase is first discretized using a spatial discretization method with local conservation, and then the discretized volume conservation equations for all phases are summed.

In [19], an IMPES scheme is developed for simulating two-phase flow in porous media, taking into account dynamic capillary pressure. The pressure equation is treated implicitly together with the saturation equation. A stability condition for the scheme is determined.

The results show that the IMPES scheme with dynamic capillary pressure is more stable than the scheme with static capillary pressure. Furthermore, unlike in the case of static capillary pressure, the IMPES scheme with dynamic capillary pressure remains stable for larger time steps.

In [20], a method for the efficient solution of multiphase flow problems in the near-wellbore region of porous media is presented. The numerical discretization is based on the IMPES approach. Therefore, parametric models with history matching are used in this region. However, parametric models contain a number of uncertainties that may affect the estimation of pressure drop. Consequently, accurate and fast simulation of this area is required. In this work, a new and reliable implementation of the IMPES scheme is proposed, aimed at reducing the significant computational cost of the (implicit) pressure system and enhancing the stability and robustness of the scheme. The method employs small physical time steps between iterations observed in the near-wellbore region to obtain a new pressure field using an explicit numerical scheme. Two- and three-dimensional numerical experiments are conducted to validate the proposed discretization. Numerical results show that the computational cost of the method is reduced by a factor of 7-19 compared to the classical IMPES scheme.

The conducted literature review shows that the IMPES method requires the implementation of an additional procedure to ensure stability [21].

S. Cengizci and Ö. Uğur [22] investigated singularly perturbed convection-diffusion problems and stabilized the standard Galerkin formulation using the streamline-upwind/Petrov-Galerkin (SUPG) approach. These studies are relevant to the nonlinear solution procedure and the treatment of convection-dominated terms in the present model, although they

do not address nonequilibrium two-phase filtration.

Stable temporal and spatial discretizations have also been developed for related transport problems. S. Thomas and S. K. Nadupuri [23] constructed a Crank–Nicolson-type method with a fourth-order compact exponential spatial approximation for a variable-order time-fractional mobile-immobile advection-dispersion model and established unique solvability and unconditional stability for that scheme [24-27].

In the context of oil-water displacement, B. A. Suleimanov and E. M. Abbasov [28] investigated pressure build-up in a flat-radial reservoir and developed an expression for estimating water breakthrough time. Their work provides an application-oriented context for relating pressure evolution to the movement of the water-oil contact.

Other numerical studies address complementary aspects of nonlinear approximation and verification. M. Izadi, H. M. Srivastava and W. Adel [29] combined quasilinearization with Bessel-polynomial-based approximation for a one-dimensional steady nonlinear reactive transport model, assessing accuracy through residual errors and benchmark problems.

Recent hydrodynamic studies also complement the fictitious-domain and splitting approaches discussed above. A. N. Temirbekov, L. N. Temirbekova and B. T. Zhumagulov [30] developed a fictitious domain algorithm based on a Lagrange functional and conjugate optimization, first for the Burgers equation and then for the nonlinear Navier–Stokes equations. B. Zhumagulov, N. Temirbekov and D. Tamabay [31] addressed the stability and convergence of a modified three-step splitting scheme for the Navier–Stokes equations. These works provide a broader numerical hydrodynamics context; the present study focuses instead on the coupled discretization of pressure, saturation and the nonequilibrium variable in the Muskat–Leverett model.

A more direct connection with porous-medium flow is provided by A. Ibragimov et al. [32], who revisited the Peaceman well-block radius and the relationship between grid-block pressure and well pressure for linear Darcy and nonlinear Forchheimer flows. Their analysis links grid-scale material balance with radial-flow pressure relations and shows that nonlinear flow modifies the effective well-block radius. These results emphasize the role of consistent near-well pressure representation in reservoir simulation and complement the present focus on the discretization of the coupled filtration equations.

This article is devoted to a further study of the Muskat–Leverett model taking into account dynamic (nonequilibrium) capillary pressure. In the nonequilibrium Muskat–Leverett model, an additional term is introduced to describe the nonequilibrium process of capillary pressure. A variational formulation of the nonequilibrium Muskat–Leverett model is constructed in a mixed form with an additional variable representing dynamic capillary pressure.

A mixed finite element scheme has been developed and numerically implemented for the nonequilibrium Muskat–Leverett model, taking into account the dynamic effect of capillary pressure. The influence of the relaxation parameter on the variations of pressure and saturation solutions is investigated through computational experiments. The program code is implemented on the FEniCS computational platform.

This article is organized as follows. The Introduction provides a literature review on the Muskat–Leverett model for

nonequilibrium filtration. In Section 2, the strong formulation and the weak formulation of the problem are presented, followed by the reduction to a bilinear form of the variational formulation. Section 3 describes the implicit numerical method for solving the problem in bilinear form. In Section 4, the IMPES method is discussed. Section 5 presents the results of numerical calculations and their discussion.

2. Problem statement

A two-phase flow model is considered, taking into account dynamic capillarity and including the mass conservation equations for the two phases: oil and water [33-35].

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded domain with a Lipschitz boundary, and $J = (0, T]$ be the time interval. The unknown functions are $s(x, t)$ - the saturation of the displaced phase, $\mathbf{q}_\alpha(x, t)$ - the filtration flux, and $p_\alpha(x, t)$ - the pressure of phase $\alpha = \{o, w\}$; the subscripts denote «o» for oil and «w» for water, respectively.

Strong formulation

Problem 1. Find the functions $(s, \mathbf{q}_o, \mathbf{q}_w, p_o, p_w)$ in $\Omega \times J$, satisfying the system

$$m \frac{\partial s}{\partial t} + \nabla \cdot \mathbf{q}_o = f_o(x, t, s), \quad (1)$$

$$m \frac{\partial (1-s)}{\partial t} + \nabla \cdot \mathbf{q}_w = f_w(x, t, s), \quad (2)$$

$$\mathbf{q}_o = -\frac{k}{\mu_o} k_o(s) \nabla p_o, \quad (3)$$

$$\mathbf{q}_w = -\frac{k}{\mu_w} k_w(s) \nabla p_w, \quad (4)$$

and the equation relating the phase pressures p_o, p_w and saturation s

$$p_o - p_w = p_c(s) + \tau \frac{\partial s}{\partial t}, \quad (5)$$

where p_c is the capillary pressure, τ is the displacement time, $m(x)$ is the porosity, and f_α are the source and sink functions.

The system (1)-(5) is supplemented with the following initial and boundary conditions:

$$s(x, 0) = s_0, \quad x \in \bar{\Omega} \quad (6)$$

$$p_o = 0, \quad p_w = 0 \quad \text{on } \partial\Omega \times J \quad (7)$$

In further studies, the following assumptions are adopted:

Assumption 2.1 (Porosity)

$$m \in L_\infty(\Omega), \quad 0 < m_* \leq m(x) \leq m^*$$

Assumption 2.2 (Relative permeabilities)

The functions $k_\alpha(s)$ are Lipschitz-continuous, with $k_o(s)$ being a non-decreasing function and $k_w(s)$ a non-increasing function, satisfying the conditions $k_* \leq k_\alpha^{-1}(s) \leq k^*$.

Assumption 2.3 (Capillary pressure)

p_c is a monotonically increasing function and $p_c \in C^1(R)$, with $p_c(s_0) = 0$. There exist constants $m_p, M_p > 0$ such that $m_p \leq p'_c(\cdot) \leq M_p < \infty$.

Assumption 2.4 (Sources)

Assume that

$$f_\alpha(x, t, s) = f_{\alpha_1}(s) + f_{\alpha_2}(x, t)$$

where there exists a constant $f^* > 0$ such that $|f_{\alpha_1}(s)| \leq C|s|$.

The article uses standard notations from functional analysis and the theory of partial differential equations. We introduce the functional spaces $W=L^2(\Omega)$, $\mathbf{V}=H(\text{div};\Omega) = \{v \in L_2(\Omega) : \nabla \cdot v \in L_2(\Omega)\}$, $Q=L_2(\Omega)$.

The weak formulation of problem (1)–(7) consists in finding the functions $s: J \rightarrow W$, $\mathbf{q}_\alpha: J \rightarrow V$, $p_\alpha: J \rightarrow Q$, $\alpha \in \{o, w\}$, such that $s \in L_2(0, T; W)$, $\frac{\partial s}{\partial t} \in L_2(0, T; W)$, $q_\alpha \in L_2(0, T; V)$, $p_\alpha \in L_2(0, T; Q)$, satisfying, for almost all $t \in (0, T)$ and for all test functions $\varphi \in W$, $\mathbf{v}_o, \mathbf{v}_w \in V$, $r \in Q$ the following identities:

$$\left(m \frac{\partial s}{\partial t}, \varphi_o\right) + (\nabla \cdot \mathbf{q}_o, \varphi_o) = (f_o(x, t, s), \varphi_o), \quad (8)$$

$$-\left(m \frac{\partial s}{\partial t}, \varphi_w\right) + (\nabla \cdot \mathbf{q}_w, \varphi_w) = (f_w(x, t, s), \varphi_w), \quad (9)$$

$$(k_o^{-1}(s) \mathbf{q}_o, \mathbf{v}_o) - (p_o, \nabla \cdot \mathbf{v}_o) = 0, \quad (10)$$

$$(k_w^{-1}(s) \mathbf{q}_w, \mathbf{v}_w) - (p_w, \nabla \cdot \mathbf{v}_w) = 0, \quad (11)$$

$$(p_o - p_w, r) = (p_c(s), r) + \tau \left(\frac{\partial s}{\partial t}, r \right) \quad (12)$$

Let T_h be a quasi-uniform triangulation of the domain Ω , with h denoting the mesh diameter. We introduce the finite element spaces $W_h \subset W$, $V_h \subset V$, $Q_h \subset Q$.

These discrete spaces are defined as follows:

$$W_h := \{p \in L_2(\Omega) / p \text{ is constant on each } T \in T_h\},$$

$$V_h := \{\mathbf{q} \in H(\text{div}; \Omega) / \mathbf{q}|_K(x) = \bar{a} + b x, \bar{a} \in \mathbb{R}^d, b \in \mathbb{R} \text{ for all } K \in T_h\}$$

We introduce a projector for the fluxes

$$\Pi_h: H(\text{div}; \Omega) \cap H^{l+1}(\Omega) \rightarrow V_h,$$

defined via moment conditions. For a face $F \subset \partial K$ of each element $K \in T_h$

$$\int_F (\mathbf{v} - \Pi_h \mathbf{v}) \mathbf{n}_F q ds = 0, \quad \forall q \in P_l(F),$$

$$\int_K (\mathbf{v} - \Pi_h \mathbf{v}) r dx = 0, \quad \forall r \in P_{l-1}(K)$$

where $P_l(F)$ is a polynomial of degree l on the face, and $P_{l-1}(K)$ is a polynomial of degree at most $l-1$ on the element. For pressure and saturation, a standard L_2 projector is used. For the pressure, $P_h: L_2(\Omega) \rightarrow Q_h$, is defined such that

$$(p - P_h p, q) = 0, \quad \forall q \in Q_h$$

The same operator is used for the saturation

$$\mathcal{P}_h: L_2(\Omega) \rightarrow W_h$$

Under sufficient smoothness of the solutions, the fluxes, pressure, and saturation satisfy the following approximation properties:

$$\|\mathbf{v} - \Pi_h \mathbf{v}\| \leq Ch^{l+1} \|\mathbf{v}\|_{l+1},$$

$$\|\nabla \cdot (\mathbf{v} - \Pi_h \mathbf{v})\| \leq Ch^{l+1} \|\nabla \cdot \mathbf{v}\|_{l+1},$$

$$\|p - P_h p\| \leq Ch^{l+1} \|p\|_{l+1},$$

$$\|s - \mathcal{P}_h s\| \leq Ch^{l+1} \|s\|_{l+1}$$

For time discretization, we define the time step $\Delta t = T/N$, as well as $t^n = n \cdot \Delta t$, $n = 0, 1, 2, \dots, N$.

In the article by X. Cao and K. Mitra [7], a semi-discrete variational formulation of problem (8)–(12) is first construct-

ed. The existence of a solution to this problem is proven, and the uniqueness of the solution is established in Lemma 1. Subsequently, a fully discrete variational formulation of problem (8)–(12) is constructed as follows:

Let $n \in \{1, 2, \dots, N\}$ and $s_h^{n-1} \in W_h$.

Find $(s_h^n, \mathbf{q}_{oh}^n, \mathbf{q}_{wh}^n, p_{oh}^n, p_{wh}^n) \in W_h \times V_h \times V_h \times Q_h \times Q_h$ such that they satisfy the following relations

$$\left(m \frac{s_h^n - s_h^{n-1}}{\Delta t}, \varphi_h\right) + (\nabla \cdot \mathbf{q}_{oh}^n, \varphi_h) = (f_{oh}^n, \varphi_h), \quad (8')$$

$$-\left(m \frac{s_h^n - s_h^{n-1}}{\Delta t}, \varphi_h\right) + (\nabla \cdot \mathbf{q}_{wh}^n, \varphi_h) = (f_{wh}^n, \varphi_h), \quad (9')$$

$$(k_o^{-1}(s_h^n) \mathbf{q}_{oh}^n, \mathbf{v}_h) - (p_{oh}^n, \nabla \cdot \mathbf{v}_h) = 0, \quad (10')$$

$$(k_w^{-1}(s_h^n) \mathbf{q}_{wh}^n, \mathbf{v}_h) - (p_{wh}^n, \nabla \cdot \mathbf{v}_h) = 0, \quad (11')$$

$$(p_o^n - p_w^n, r_h) = (p_c(s_h^n), r_h) + \tau \left(\frac{s_h^n - s_h^{n-1}}{\Delta t}, r_h \right), \quad (12')$$

for all $\varphi_h \in W_h$, $\mathbf{v}_h \in V_h$, $r_h \in Q_h$. At $t=0$, we take $s_h^0 = P_h(s^0)$.

In [7], the convergence of the solution of the discrete problem (8')–(12') to the solution of problem (8)–(12) is proven. To this end, we introduce the following notations:

$$e_s^n = s(t^n) - s_h^n,$$

$$e_{q_o}^n = \mathbf{q}_o(t^n) - \mathbf{q}_{oh}^n,$$

$$e_{q_w}^n = \mathbf{q}_w(t^n) - \mathbf{q}_{wh}^n,$$

$$e_{p_o}^n = p_o(t^n) - p_{oh}^n,$$

$$e_{p_w}^n = p_w(t^n) - p_{wh}^n$$

The result of the proof is formulated as the following theorem:

Theorem 1. Let $(s_h^n, \mathbf{q}_{oh}^n, \mathbf{q}_{wh}^n, p_{oh}^n, p_{wh}^n)$, $n \in \{1, 2, \dots, N\}$, be the solution of problem (8')–(12'), and assume that Assumptions (2.1)–(2.4) hold. Then, for a sufficiently small Δt , we have for any $n \in \{1, 2, \dots, N\}$:

$$\|e^n\|^2 = \|e_s^n\|^2 + \|e_{q_o}^n\|^2 + \|e_{q_w}^n\|^2 + \|e_{p_o}^n\|^2 + \|e_{p_w}^n\|^2 \leq C(\Delta t^2 + h^2)$$

with a constant C independent of Δt and h .

Next, in order to extract the bilinear form and reveal the saddle-point structure, we rewrite the system (8)–(12) in operator form.

We introduce the following notations:

$$A_\alpha(s; \mathbf{q}, \mathbf{v}) = (k_\alpha^{-1}(s), \mathbf{q}, \mathbf{v}),$$

$$B(\mathbf{q}, \varphi) = (\nabla \cdot \mathbf{q}, \varphi)$$

and rewrite (8)–(12) in the form:

$$\left(m \frac{\partial s}{\partial t}, \varphi_o\right) + A_o(s; \mathbf{q}_o, \mathbf{v}_o) + B(\mathbf{q}_o, \varphi_o) - B(\mathbf{v}_o, p_o) = (f_o(x, t, s), \varphi_o), \quad (13)$$

$$-\left(m \frac{\partial s}{\partial t}, \varphi_w\right) + A_w(s; \mathbf{q}_w, \mathbf{v}_w) + B(\mathbf{q}_w, \varphi_w) - B(\mathbf{v}_w, p_w) = (f_w(x, t, s), \varphi_w), \quad (14)$$

$$\left(\frac{\partial s}{\partial t}, r\right) + \frac{1}{\tau} (p_c(s), r) = \frac{1}{\tau} (p_o - p_w, r) \quad (15)$$

By summing (13) and (14), we arrive at the weak formulation of problem (1)–(7).

Problem 2. Find $(s, \mathbf{q}_o, \mathbf{q}_w, \mathbf{p}_o, \mathbf{p}_w) : J \rightarrow W \times V \times V \times Q \times Q$, satisfying the following identities:

$$\left(m \frac{\partial s}{\partial t}, \varphi_o - \varphi_w \right) + \sum_{\alpha \in \{o, w\}} (A_\alpha(s; \mathbf{q}_\alpha, \mathbf{v}_\alpha) + B(\mathbf{q}_\alpha, \varphi_\alpha) - B(\mathbf{v}_\alpha, p_\alpha)) = \sum_{\alpha \in \{o, w\}} ((f_{\alpha_1}(s), \varphi_\alpha) + (f_{\alpha_2}, \varphi_\alpha)), \quad (16)$$

$$\left(\frac{\partial s}{\partial t}, r \right) + \frac{1}{\tau} (p_c(s), r) = \frac{1}{\tau} (p_o - p_w, r) \quad (17)$$

for all $(\varphi_o, \varphi_w, \mathbf{v}_o, \mathbf{v}_w, r) \in W \times V \times V \times Q \times Q$.

The form of the integral identities (16) and (17) is important for a compact representation that is symmetric with respect to the phases and convenient for obtaining energy estimates. The main advantage is that it reveals the saddle-point structure. In this representation:

$$A_\alpha(s; \mathbf{q}_\alpha, \mathbf{v}_\alpha) + B(\mathbf{q}_\alpha, \varphi_\alpha) - B(\mathbf{v}_\alpha, p_\alpha)$$

the saddle-point structure of the type

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \quad (18)$$

becomes clearly visible.

This is the classical structure of a mixed problem of the Darcy-Stokes type. From (16) and (17), we can derive an energy estimate. To this end, we set $\mathbf{v}_\alpha = \mathbf{q}_\alpha$, $\varphi_\alpha = p_\alpha$, $r = s$ and substitute into (16):

$$\begin{aligned} & \left(m \frac{\partial s}{\partial t}, p_o - p_w \right) + \\ & + \sum_{\alpha \in \{o, w\}} (A_\alpha(s; \mathbf{q}_\alpha, \mathbf{q}_\alpha) + B(\mathbf{q}_\alpha, p_\alpha) - B(\mathbf{q}_\alpha, p_\alpha)) \\ & = \sum_{\alpha \in \{o, w\}} ((f_{\alpha_1}(s), p_\alpha) + (f_{\alpha_2}, p_\alpha)) \end{aligned} \quad (19)$$

By definition $A_\alpha(s; \mathbf{q}_\alpha, \mathbf{q}_\alpha) = (k_\alpha^{-1}(s), \mathbf{q}_\alpha, \mathbf{q}_\alpha)$ and by the assumption $k_* \leq k_\alpha^{-1}(s) \leq k^*$, it follows that

$$(k_\alpha^{-1}(s), \mathbf{q}_\alpha, \mathbf{q}_\alpha) \geq k_* \|\mathbf{q}_\alpha\|^2 \quad (20)$$

The terms $B(\mathbf{q}_\alpha, p_\alpha)$ cancel out completely.

Substituting expression (5) into the first term of (19), we obtain

$$\left(m \frac{\partial s}{\partial t}, p_o - p_w \right) = m \tau \cdot \left(\frac{\partial s}{\partial t}, \frac{\partial s}{\partial t} \right) + \left(\frac{\partial s}{\partial t}, p_c(s) \right) \quad (21)$$

Combining (19) and (21), we obtain the energy equality

$$\tau m \left\| \frac{\partial s}{\partial t} \right\|^2 + \left(\frac{\partial s}{\partial t}, p_c(s) \right) + \sum_\alpha (k_\alpha^{-1}(s), \mathbf{q}_\alpha, \mathbf{q}_\alpha) = \sum_\alpha (f_{\alpha_2}, p_\alpha) \quad (22)$$

We transform the term with the capillary pressure, since $p_c(s)$ is an increasing function:

$$\left(\frac{\partial s}{\partial t}, p_c(s) \right) = \frac{\partial}{\partial t} \int_\Omega \Pi(s) dx, \quad (23)$$

where

$$\Pi(s) = \int_0^s p_c(\xi) d\xi$$

The final energy formula takes the form:

$$\frac{\partial}{\partial t} \int_\Omega \Pi(s) dx + \tau m \left\| \frac{\partial s}{\partial t} \right\|^2 + \sum_\alpha (k_\alpha^{-1}(s), \mathbf{q}_\alpha, \mathbf{q}_\alpha) = \sum_\alpha (f_{\alpha_2}, p_\alpha) \quad (24)$$

The right-hand side is estimated using the Cauchy-Bunyakovsky inequality:

$$(f_\alpha, p_\alpha) \leq \frac{1}{2\varepsilon} \|f_\alpha\|^2 + \frac{\varepsilon}{2} \|p_\alpha\|^2 \quad (25)$$

The pressure p_α and flux $\mathbf{q}_\alpha$ are related by Darcy's law:

$$\mathbf{q}_\alpha = -k_\alpha(s) \nabla p_\alpha$$

Using the inverse Poincaré inequality for the pressure with boundary conditions $p_\alpha|_{\partial\Omega} = 0$, we obtain:

$$\|p_\alpha\| \leq C_p \|\nabla p_\alpha\| = C_p \|k_\alpha^{-1}(s) \mathbf{q}_\alpha\| \leq C_p k^* \|\mathbf{q}_\alpha\| \quad (26)$$

where C_p is the Poincaré constant and k^* is the upper bound $k_\alpha^{-1}(s)$.

Thus,

$$\frac{\varepsilon}{2} \|p_\alpha\|^2 \leq \frac{\varepsilon}{2} (C_p k^*)^2 \|\mathbf{q}_\alpha\|^2$$

We choose $\varepsilon > 0$ sufficiently small so that

$$\frac{\varepsilon}{2} (C_p k^*)^2 \leq \frac{k_*}{2}$$

Then the left-hand side of the energy estimate contains

$$(k_\alpha^{-1}(s), \mathbf{q}_\alpha, \mathbf{q}_\alpha) - \frac{\varepsilon}{2} \|p_\alpha\|^2 \geq \frac{k_*}{2} \|\mathbf{q}_\alpha\|^2 > 0 \quad (27)$$

Then, from (20), (24), (25), and (27), we obtain

$$\frac{\partial}{\partial t} \int_\Omega \Pi(s) dx + \tau m \left\| \frac{\partial s}{\partial t} \right\|^2 + \frac{k_*}{2} \sum_\alpha \|\mathbf{q}_\alpha\|^2 \leq C \sum_\alpha \|f_\alpha\|^2 \quad (28)$$

Integrating inequality (28) over t , we obtain

$$\begin{aligned} & \int_\Omega \Pi(s(t)) dx + \tau m \int_0^t \left\| \frac{\partial s}{\partial t} \right\|^2 dt + \frac{k_*}{2} \sum_\alpha \int_0^t \|\mathbf{q}_\alpha\|^2 dt \leq \\ & \leq C \sum_\alpha \int_0^t \|f_\alpha\|^2 dt, \end{aligned}$$

The final energy estimate reads:

$$\sup_{t \in (0, T)} \int_\Omega \Pi(s(t)) dx + \int_0^T \left\| \frac{\partial s}{\partial t} \right\|^2 dt + \int_0^T \|\mathbf{q}_\alpha\|^2 dt \leq C \quad (29)$$

Estimate (29) provides bounds on the saturation, bounds on the fluxes, control of the time derivative, existence of a weak solution, and stability of the finite element method.

3. Implicit numerical method for the implementation of problem (16), (17)

In the nonequilibrium Muskat-Leverett model, the fully implicit scheme (Fully Implicit Method, FIM) has several fundamental advantages compared to the IMPES method (Implicit Pressure, Explicit Saturation), especially for numerical implementation in the FEniCS environment. The main advantage of the fully implicit scheme is unconditional stability with respect to the time step. The second significant advantage is improved convergence and stability under strong nonlinearities. From the perspective of implementation in FEniCS, the fully implicit approach also fits naturally into the variational formulation: the entire system of equations is formulated in weak form and solved as a single nonlinear problem. This simplifies the use of automatic differentiation for constructing the Jacobian and makes the code more compact and mathematically consistent.

Problem 3 (Discrete problem).

For each time step, find $(s_h^{n+1}, \mathbf{q}_{oh}^{n+1}, \mathbf{q}_{wh}^{n+1}, p_{oh}^{n+1}, p_{wh}^{n+1}) \in W_h \times V_h \times V_h \times Q_h \times Q_h$ with $s_h^0 = p_h^0$, such that

$$\begin{aligned} & \left(m \frac{s_h^{n+1} - s_h^n}{\Delta t}, \varphi_{oh} - \varphi_{wh} \right) + \\ & + \sum_{\alpha \in \{o, w\}} A_\alpha \left(s_h^{n+1}; \mathbf{q}_{\alpha h}^{n+1}, \mathbf{v}_{\alpha h} \right) + B \left(\mathbf{q}_{\alpha h}^{n-1}, \varphi_{\alpha h} \right) - B \left(\mathbf{v}_{\alpha h}, p_{\alpha h}^{n+1} \right) = \\ & = \sum_{\alpha \in \{o, w\}} \left(\left(f_{\alpha_1} \left(s_h^{n+1} \right), \varphi_{\alpha h} \right) + \left(f_{\alpha_2}, \varphi_{\alpha h} \right) \right), \end{aligned} \quad (30)$$

$$\left(\frac{s_h^{n+1} - s_h^n}{\Delta t}, r_h \right) + \frac{1}{\tau} \left(p_c \left(s_h^{n+1} \right), r_h \right) = \frac{1}{\tau} \left(p_{oh}^{n+1} - p_{wh}^{n+1}, r_h \right) \quad (31)$$

for all $(r_h, \mathbf{v}_{oh}, \mathbf{v}_{wh}, \varphi_{oh}, \varphi_{wh}) \in W_h \times \mathbf{V}_h \times \mathbf{V}_h \times Q_h \times Q_h$.

At each time step, a nonlinear system arises:

$$\begin{pmatrix} M_s & 0 & 0 & 0 & 0 \\ 0 & A_o(s^{n+1}) & 0 & -B^T & 0 \\ 0 & 0 & A_w(s^{n+1}) & 0 & -B^T \\ 0 & B & 0 & 0 & 0 \\ 0 & 0 & B & 0 & 0 \end{pmatrix} \begin{pmatrix} s^{n+1} \\ q_o^{n+1} \\ q_w^{n+1} \\ p_o^{n+1} \\ p_w^{n+1} \end{pmatrix} = RHS \quad (32)$$

where M_s is the mass matrix for the saturation:

$$(M_s)_{ij} = (m \phi_j, \phi_i),$$

$A_\alpha(s^{n+1})$ is the mobility matrix (depending on s^{n+1}), B is the divergence matrix: $(B)_{ij} = (\nabla \cdot \mathbf{v}_j, \psi_i)$, the vector of unknowns is $\mathbf{U}^{n+1} = (s^{n+1}, q_o^{n+1}, q_w^{n+1}, p_o^{n+1}, p_w^{n+1})^T$.

The right-hand side is

$$RHS = \begin{pmatrix} \frac{1}{\Delta t} M_s s^n + F_o^{n+1}(s^{n+1}) + F_w^{n+1}(s^{n+1}) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (33)$$

where $F_\alpha^{n+1}(s^{n+1}) = \left((f_{\alpha_1}(s^{n+1}), \phi_i) + f_{\alpha_2}, \phi_i \right)$.

This system $G(\mathbf{U}) = 0$ is solved using the Newton method:

$$\begin{aligned} G(\mathbf{U}^k + \delta \mathbf{U}) & \approx G(\mathbf{U}^k) + J(\mathbf{U}^k) \times \delta \mathbf{U} \\ J(\mathbf{U}^k) \times \delta \mathbf{U} & = -G(\mathbf{U}^k) \end{aligned}$$

where $J(\mathbf{U}^k) = G'(\mathbf{U}^k)$ is the Jacobian matrix (matrix of partial derivatives) with entries $J_{ij}(\mathbf{U}^k) = \frac{\partial G_i}{\partial U_j}(\mathbf{U}^k)$.

From (34), $\delta \mathbf{U}$ is computed, and the next approximation is updated as

$$\mathbf{U}^{k+1} = \mathbf{U}^k + \delta \mathbf{U} \quad (35)$$

with iterations continuing until

$$\|G(\mathbf{U}^{k+1})\| \leq \varepsilon \quad (36)$$

4. Classical method for solving the Muskat-Leverett model in pressure-saturation variables

The Implicit Pressure – Explicit Saturation (IMPES) method is a classical baseline approach for two-phase flow. We demonstrate that our fully implicit FIM is more stable, performs better under nonequilibrium conditions, and does not require a strict time-step restriction.

In the nonequilibrium model, where an additional relax-

ation (kinetic) term appears describing the dynamics of capillary pressure, this time-step restriction becomes even more stringent. The IMPES method is sensitive to the time step and can lose stability for large τ .

In IMPES, splitting by physical variables can lead to error accumulation, oscillations, and even instability under large contrasts in phase properties. The IMPES method requires manual splitting of the system and careful management of data transfer between time steps, which complicates the program structure.

Below, we provide a description of the IMPES method.

1. Correct formulation of IMPES.

Pressure – implicit:

$$\begin{aligned} \nabla \cdot \left(k_w(s^n) (\nabla p_w^{n+1} - \rho_w \vec{g}) + k_o(s^n) (\nabla (p_o^{n+1} + \pi^n) - \rho_o \vec{g}) \right) = \\ = f_w^{n+1} + f_o^{n+1} \end{aligned}$$

Saturation – explicit:

$$\frac{s^{n+1} - s^n}{\tau} + \nabla \cdot \mathbf{q}_w^{n+1} = f_w^{n+1}$$

where $\mathbf{q}_w^{n+1} = k_w(s^n) (\nabla p_w^{n+1} - \rho_w \vec{g})$.

Nonequilibrium – explicit:

$$\pi^{n+1} = s^n + \tau \cdot \frac{s^n - s^{n-1}}{\Delta t}$$

Algorithm for implementing the IMPES method:

1. Set s^0 – the initial saturation values.
2. Compute the coefficients $k_w(s^n)$, $k_o(s^n)$.
3. Solve the elliptic problem for p^{n+1} .
4. Solve the saturation equation for s^{n+1} .
5. Update the capillary pressure.

The pressure equation is solved as a variational problem:

$$\begin{aligned} \int_{\Omega} (k_w^n + k_o^n) \nabla p^{n+1} \cdot \nabla v \, dx = \int_{\Omega} k_o^n \nabla \pi^n \cdot \nabla v \, dx + \\ + \int_{\Omega} (k_w^n \cdot \rho_w \vec{g} + k_o^n \cdot \rho_o \vec{g}) \cdot \nabla v \, dx + \int_{\Omega} (f_w^{n+1} + f_o^{n+1}) v \, dx \end{aligned}$$

Since the saturation equation is hyperbolic, the standard FEM can produce oscillations. Therefore, the SUPG (Streamline Upwind Petrov–Galerkin) method is used. The stabilization parameter is defined as $\tau_{SUPG} = \frac{0.14}{2|\mathbf{q}_w|}$ and an additional term is added to the saturation equation:

$$\int_{\Omega} \tau_{SUPG} (\mathbf{q}_w \cdot \nabla s) (\mathbf{q}_w \cdot \nabla v) \, dx$$

5. Numerical solution and discussion of results

To illustrate the capabilities of the finite element method under consideration, the computational domain was chosen as $\Omega = (0,1) \times (0,1)$. A uniform triangular mesh was constructed. In this study, the phase fluxes account for gravity, which is directed along the x -axis, and are defined as:

$$\mathbf{q}_\alpha = -k_\alpha(s) \cdot (\nabla p_\alpha - \rho_\alpha \cdot \vec{g}) \quad (37)$$

The functions k_o , k_w and p_c were taken in the calculations as

$$k_o(s) = s, \quad k_w(s) = 1 - s, \quad p_c(s) = s \quad (38)$$

The initial and boundary conditions are specified as follows:

$$\begin{aligned}
 \text{at } t = 0, \quad s(x, y, 0) &= 1 - \frac{1}{4}(1 + x^2 + y^2), \\
 \text{at } x = 0, \quad p_o(0, y, t) &= 1 + \frac{\tau}{4}(1 + y^2) \cdot e^{-t}, \\
 p_w(0, y, t) &= \frac{1}{4}(1 + y^2) \cdot e^{-t}, \\
 \text{at } x = 1, \quad p_o(1, y, t) &= 1 + \frac{\tau}{4}(2 + y^2) \cdot e^{-t}, \\
 p_w(1, y, t) &= \frac{1}{4}(2 + y^2) \cdot e^{-t}, \\
 \text{at } y = 0, \quad \bar{q}_{oy} &= 0, q_{wy} = 0, \\
 \text{at } y = 1, \quad \mathbf{q}_{oy} &= -\frac{\tau}{2} \left(1 - \frac{1}{4}(2 + x^2) \cdot e^{-t} \right) \cdot e^{-t} \\
 \mathbf{q}_{wy} &= -\frac{1}{8}(2 + x^2) \cdot e^{-2t}
 \end{aligned} \quad (39)$$

The source terms were chosen as functions of spatial variables and time:

$$\begin{aligned}
 f_o &= \frac{e^{-t}}{4} \left[1 + x^2 + y^2 - 2\rho_o x + \tau \cdot \left((1 + 2x^2 + 2y^2) \cdot e^{-t} - 4 \right) \right], \\
 f_w &= -\frac{e^{-t}}{4} \left[1 + x^2 + y^2 - 2\rho_w x + (1 + 2x^2 + 2y^2) \cdot e^{-t} \right]
 \end{aligned} \quad (40)$$

Next, the weak formulations of the equations are constructed.

Oil equation:

$$\begin{aligned}
 \int_{\Omega} \frac{s^{n+1} - s^n}{\Delta t} \cdot v d\Omega + \int_{\Omega} k_o(s^{n+1}) \cdot (\nabla p_o^{n+1} - \rho_o \hat{g}) \nabla v d\Omega = \\
 = \int_{\Omega} f_o \cdot v d\Omega
 \end{aligned} \quad (41)$$

Water equation:

$$\begin{aligned}
 \int_{\Omega} \frac{s^{n+1} - s^n}{\Delta t} \cdot w d\Omega + \int_{\Omega} k_w(s^{n+1}) \cdot (\nabla p_w^{n+1} - \rho_w \hat{g}) \nabla w d\Omega = \\
 = \int_{\Omega} f_w \cdot w d\Omega,
 \end{aligned} \quad (42)$$

Dynamic capillary pressure equation:

$$\int_{\Omega} \left(p_o - p_w - p_c(s) - \tau \frac{\partial s}{\partial t} \right) v_p d\Omega = 0 \quad (43)$$

By summing equations (41), (42), and (43), a monolithic nonlinear system is obtained.

During testing, if in equations (41), (42), and (43) we choose $v = p_o$, $w = p_w$ and $v_p = \partial s / \partial t$, we obtain the discrete analog of the energy estimate:

$$\frac{d}{dt} \left(\|s\|^2 + \tau \left\| \frac{\partial s}{\partial t} \right\|^2 \right) + \|\nabla p_o\|^2 + \|\nabla p_w\|^2 \leq C \cdot (\|f_o\|^2 + \|f_w\|^2) \quad (44)$$

This estimate explains why the scheme remains stable for large τ and why the monolithic approach is preferable to the IMPES method. In IMPES, pressure is treated implicitly, while saturation is treated explicitly.

The posed problem (30)-(31) was solved using the FEniCS computational platform. The nonlinear system is solved using the Newton method. The monolithic approach ensures high stability.

The Python code developed on the FEniCS platform implements the weak formulation of the Muskat-Leverett model, accounting for gravity and dynamic capillary pressure.

Our computation demonstrates:

1. Correct implementation of the nonequilibrium Muskat-Leverett model.
2. Convergence of the finite element method.
3. Accurate approximation of the additional nonequilibrium variable $\pi = s + \tau \frac{\partial s}{\partial t}$.
4. Consistency of the numerical solution with the exact analytical solution.

To assess the method's accuracy, the following errors were computed:

$$\|s - s_h\|_{L^2(\Omega)}, \|p - p_h\|_{L^2(\Omega)}, \|\pi - s\|_{L^2(\Omega)}$$

Figure 1 shows the evolution of the errors over time. The

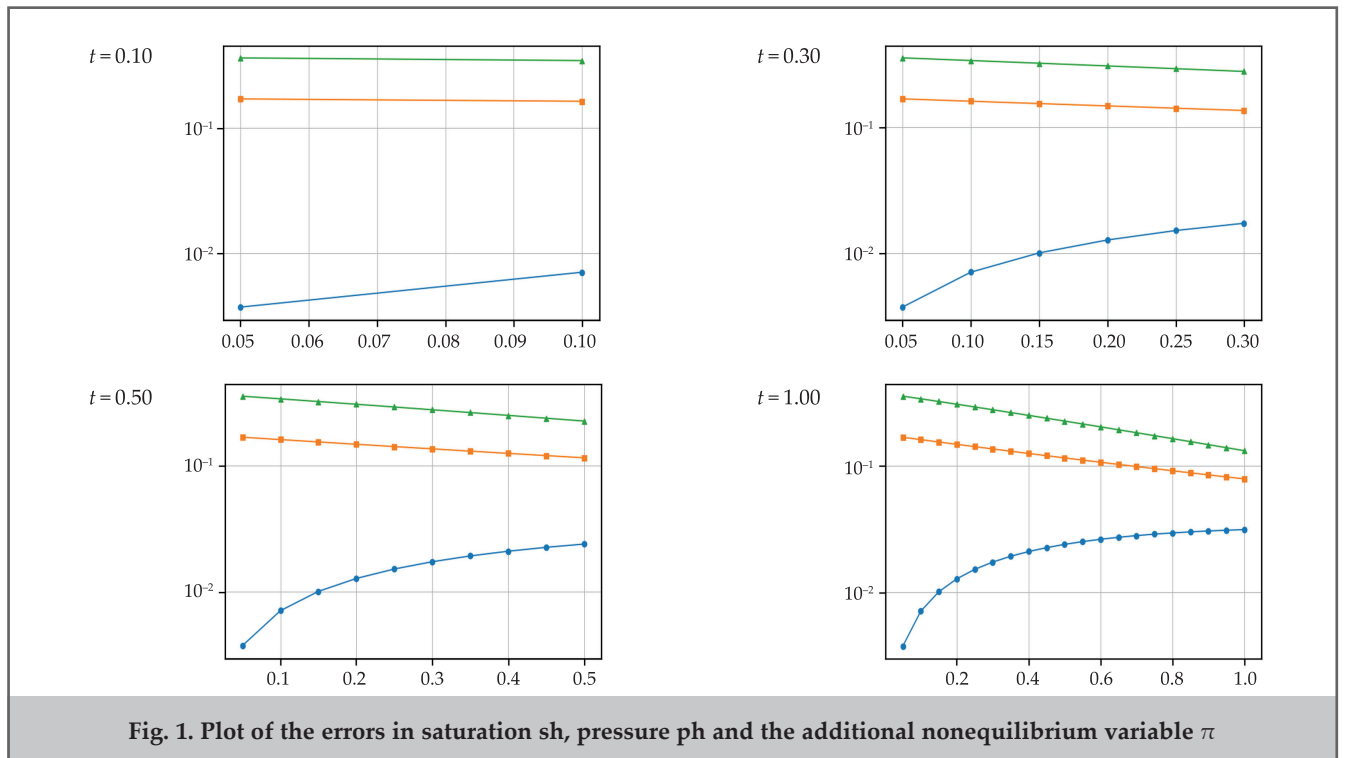


Fig. 1. Plot of the errors in saturation s_h , pressure p_h and the additional nonequilibrium variable π

numerical results demonstrate the expected behavior and good agreement with the exact solution. In particular, the nonequilibrium variable π converges to the saturation s , confirming the consistency of the relaxation model.

Figure 2 illustrates a three-dimensional surface plot of the saturation $s_h(x, y)$ at the final time $T=1$. Figure 3 shows the pressure plot at $T=1$.

Figures 4 - 5 show the error plots of s_h , p_h , π for different mesh sizes.

The numerical solution is smooth and reproduces the qualitative features of the exact solution. Computational experiments demonstrated mesh convergence. Calculations were performed for different values of $\tau=1; 0.1; 0.01$. The results show that as $\tau \rightarrow 0$, the model approaches the equilibrium case.

For comparison, the posed problem was also solved numerically using the IMPES method.

A test with a known analytical solution was employed:

$$s(x, y, t) = 1 - \frac{1}{4}(1 + x^2 + y^2) \cdot e^{-t},$$

$$p(x, y, t) = \frac{1}{4}(1 + x^2 + y^2) \cdot e^{-t}$$

The error is measured in the L_2 norm. An 80×80 mesh

(Unit Square Mesh) is used, with the function space $V=CG(1)$ and the standard Galerkin approximation. The implementation is carried out on the FEniCS computational platform. The pressure equation is solved as a system of linear equations.

Figure 6 shows the plots of the exact and approximate saturation values at $y=0.5$.

Figure 7 shows the saturation plot obtained using the IMPES method.

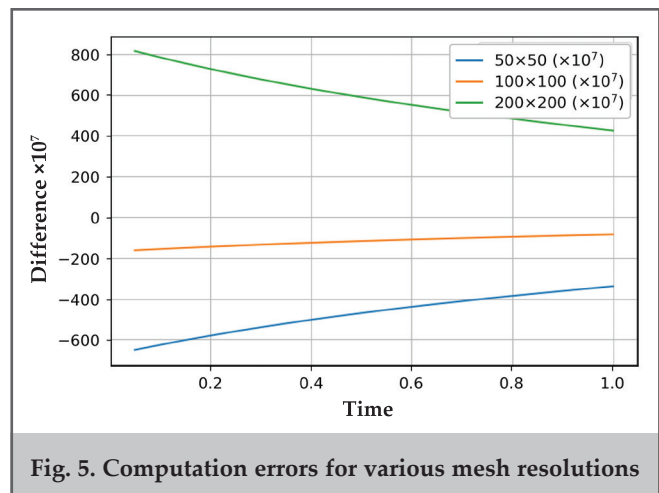
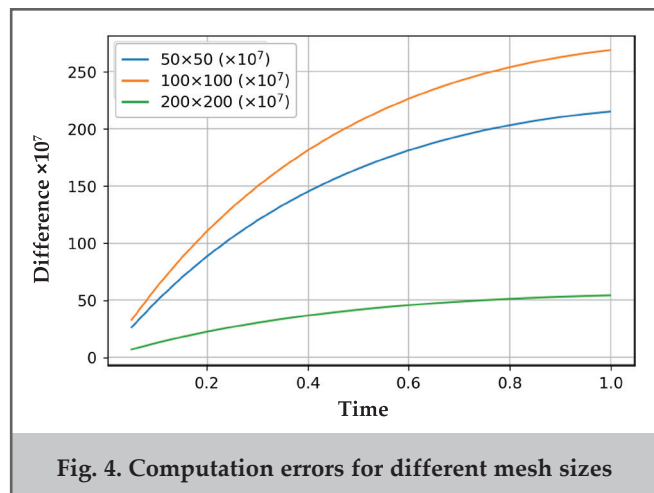
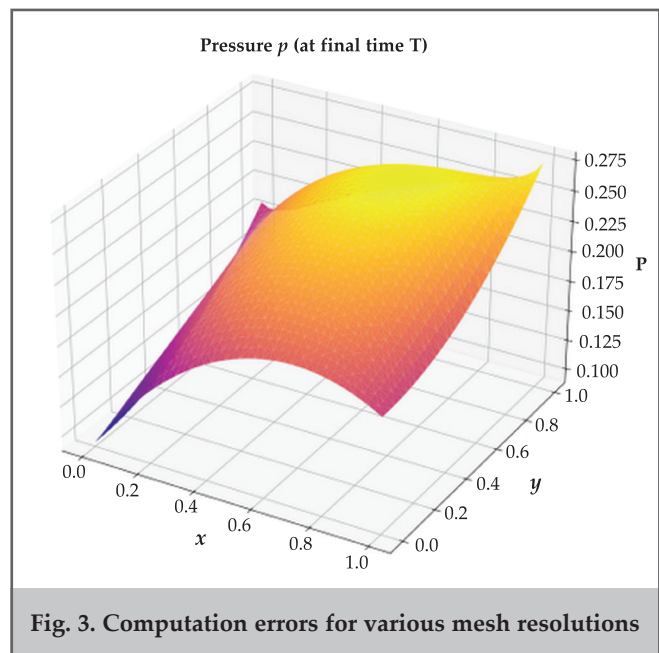
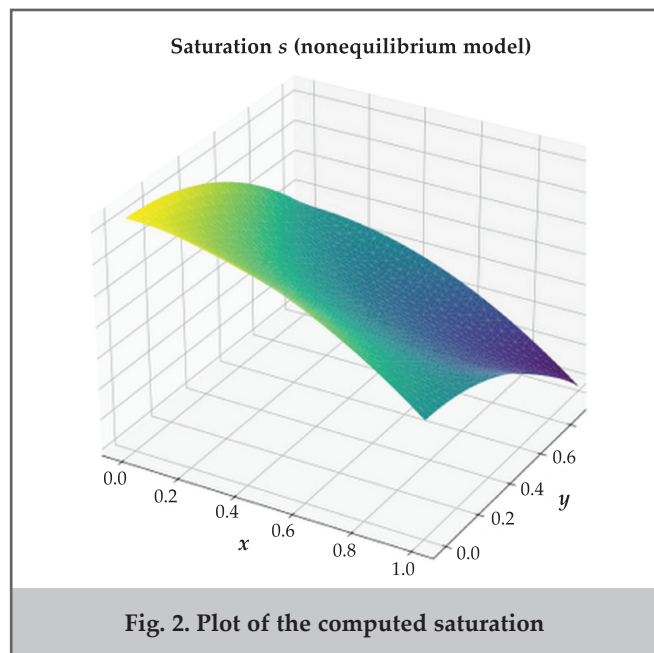
Thus, it is numerically demonstrated that the IMPES method performs poorly for the nonequilibrium filtration model.

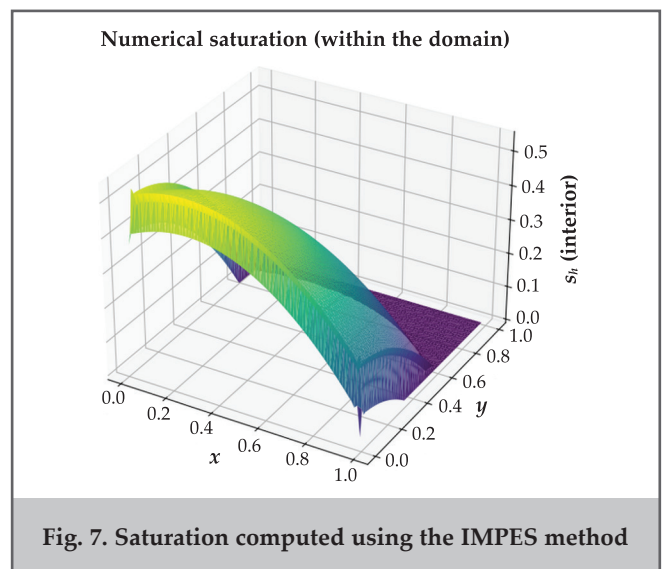
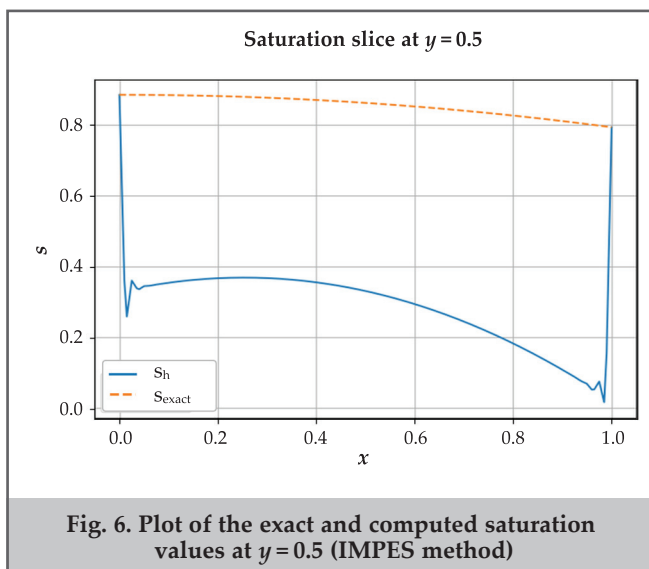
For the present computations, the errors are:

$$\|s - s_h\|_{L_2} = 0.60663564,$$

$$\|p - p_h\|_{L_2} = 0.30208661487$$

Although the IMPES scheme is widely used for equilibrium two-phase flow models, the numerical results indicate that it cannot accurately reproduce the solution obtained in the nonequilibrium case. This discrepancy arises from the explicit treatment of both the saturation and relaxation equations, leading to a loss of consistency with the continuous model. In particular, the error associated with the nonequilibrium variable remains significant even for small time steps.





Conclusions

A finite element method for the nonequilibrium Muskat–Leverett filtration model was presented and solved numerically. The proposed implicit scheme accurately reproduces the obtained solution and demonstrates stable behavior in the presence of gravity and relaxation effects. The convergence of the nonequilibrium capillary pressure to saturation confirms the model’s consistency. The proposed approach provides a solid foundation for further studies of nonequilibrium two-phase flow in porous media.

An IMPES algorithm for the nonequilibrium Muskat–Leverett model with dynamic capillary pressure was developed and implemented. It was shown that the scheme is stable for small time steps. SUPG stabilization significantly improves the solution behavior.

Comparison with the IMPES approach highlights the necessity of a fully implicit discretization for nonequilibrium two-phase flow models. Although IMPES remains efficient for equilibrium formulations, it is not suitable for capturing dynamic capillary pressure effects without stringent time-step restrictions.

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